8 April 2025

Solutions 7 : Bifurcations in Two Dimensions

1 Dulac's Criterion

In biology, a simple competitive version of the logistic model is

$$\begin{cases} \dot{x}_1 = r_1 x_1 \left(1 - \dfrac{x_1}{N_1}\right) - b_1 x_1 x_2 \\ \dot{x}_2 = r_2 x_2 \left(1 - \dfrac{x_2}{N_2}\right) - b_2 x_1 x_2 \end{cases}$$

(a) What should be the sign of the variables x_1, x_2 and parameters r_1, r_2, N_1, N_2, b_1 and b_2. Explain their physical meaning.

All are positive. The variables x_1 and x_2 are positive because they represent the number of individuals in a population. The terms proportional to r_1 and r_2 represent the logistic part of the equation, the growth is proportional to r_1 and r_2 for small populations, meaning, $r_1, r_2 > 0$. The quadratic terms with N_1 and N_2 are added to make the growth negative above $x_1 = N_1$ and $x_2 = N_2$. So $N_1, N_2 > 0$. Finally, the terms with b_1 and b_2 are added to model the competition between the two species, which decreases their growth as x_1 or x_2 get larger. We should have $b_1, b_2 > 0$.

(b) Show that there is no limit cycle for $x_1, x_2 > 0$ using the weighting function $g = \frac{1}{x_1 x_2}$.

As the title suggests, you can use Dulac's criterion to show that there are no limit cycles. This implies to check that the sign of $\nabla \cdot (g\dot{\vec{x}})$ is constant.

$$\nabla \cdot \left(\frac{1}{x_1 x_2}\begin{pmatrix}\dot{x_1} \\ \dot{x_2}\end{pmatrix}\right) = \nabla \cdot \begin{pmatrix} \dfrac{r_1}{x_2}\left(1 - \dfrac{x_1}{N_1}\right) - b_1 \\ \dfrac{r_2}{x_1}\left(1 - \dfrac{x_2}{N_2}\right) - b_2 \end{pmatrix} = -\frac{r_1}{x_2}\frac{1}{N_1} - \frac{r_2}{x_1}\frac{1}{N_2} < 0$$

The sign is strictly negative because the signs of the parameters and variables are strictly positive. The Dulac criterion says that there exist no limit cycle in the quadrant $x_1, x_2 > 0$ we are analysing.

2 Hopf Bifurcation

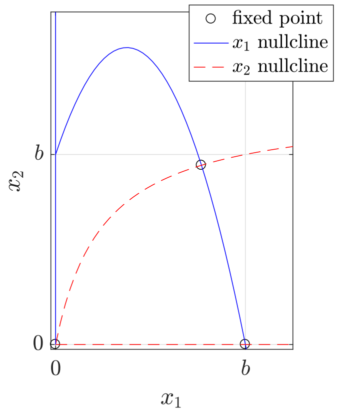
Consider the predator-prey model

$$\begin{cases} \dot{x}_1 = x_1 \left(b - x_1 - \dfrac{x_2}{1 + x_1}\right) \\ \dot{x}_2 = x_2 \left(\dfrac{x_1}{1 + x_1} - a x_2\right) \end{cases}$$

with parameters $a, b > 0$. In the first equation, the prey x_1 shows an expected exponential growth and a negative overpopulation factor $-x_1^2$. The last term is the loss due to predators, linear in x_2. If x_1 is low, $x_1 x_2/(1 + x_1) \approx x_2 x_1$, i.e. the deaths are proportional to the number of encounters between prey and predator. If the number of prey x_1 is high, $x_1 x_2/(1 + x_1) \approx x_2$, the number of deaths becomes independent of x_1 (there is a saturation as each predator is at its maximum eating capacity).

(a) Find the nullclines of the system.

To find the x_1 nullclines we solve $\dot{x}_1 = 0 = x_1(b - x_1 - \frac{x_2}{1+x_1})$, therefore $x_1 = 0$ or $b - x_1 - \frac{x_2}{1+x_1} = 0$, that is $x_2 = (1+x_1)(b-x_1)$. For the x_2 nullclines solve $\dot{x}_2 = 0 = x_2(\frac{x_1}{1+x_1} - ax_2)$ meaning that $x_2 = 0$ or $x_2 = \frac{x_1}{a(1+x_1)}$.



(b) Two fixed points lie on the x_1 and x_2 axis. Find them and, by linearising the system, determine their stability properties, if possible.

The two fixed points that lie on the x_1, x_2 axes are $(x_1^*, x_2^*) = \{(0,0), (b,0)\}$. More complicated fixed points are the intersection of $x_2 = (1+x_1)(b-x_1)$ and $x_2 = \frac{x_1}{a(1+x_1)}$. To find these points, one needs to solve the cubic equation $(1+x_1)(b-x_1) = \frac{x_1}{a(1+x_1)}$ for $x_1, x_2 > 0$ since the number of preys and predators is strictly positive.

For the linearisation, the Jacobian matrix is

$$J(x_1, x_2) = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} b - \frac{x_2}{(1+x_1)^2} - 2x_1 & \frac{-x_1}{1+x_1} \\ \frac{x_2}{(1+x_1)^2} & \frac{x_1}{1+x_1} - 2ax_2 \end{bmatrix}$$

At the origin

$$J(0,0) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$$

with eigenvalues $b > 0$ and 0. While we cannot completely assess the nature of this fixed point, we can say that it displays an unstable manifold. Also

$$J(b,0) = \begin{bmatrix} -b & \frac{-b}{1+b} \\ 0 & \frac{b}{1+b} \end{bmatrix}$$

has eigenvalues $-b < 0$ and $\frac{b}{1+b} > 0$. This is a saddle point.

(c) Prove that, for all values $a, b > 0$, there exists at least a fixed point with $x_1^*, x_2^* > 0$.

Beyond the points on the x_1 and x_2 axes, further fixed points are intersections of $x_2 = (1+x_1)(b-x_1)$ and $x_2 = \frac{x_1}{a(1+x_1)}$. Finding them requires to solve the cubic equation $(1+x_1)(b-x_1) = \frac{x_1}{a(1+x_1)}$ that can have up to 3 real solutions for $x_1 > 0$. To prove that there is at least an additional fixed point, we can use the fact that the parabola $x_2 = (1+x_1)(b-x_1)$ is continuous and goes from b at $x_1 = 0$ to 0 at $x_1 = b$, as well as the continuity of $x_2 = \frac{x_1}{a(1+x_1)}$ which goes from 0 at $x_1 = 0$ to $1/a$ as $x_1 \to \infty$. Therefore these two curves must intersect at least once in the interval $[0, b]$.

(d) A Hopf bifurcation can only occur when the trace of the linearized system changes sign. Use this fact to find the critical value $a_c(b)$ at which the Hopf bifurcation can occur at the fixed point (x_1^*, x_2^*). Is there a condition on b for the Hopf bifurcation to exist ?

The trace is

$$\mathrm{Tr}[J(x_1^*, x_2^*)] = b - \frac{x_2^*}{(1+x_1^*)^2} - 2x_1^* + \frac{x_1^*}{1+x_1^*} - 2ax_2^*$$

We need to solve a system of three equations, two equations impose that the fixed point is on both nullclines and the third equation sets the trace to zero. Since we want to find a_c as a function of b, the three unknowns are x_1^*, x_2^* and a, and b is a parameter. For the trace equation, use $x_2^* = \frac{x_1^*}{a(1+x_1^*)}$ to eliminate the $-2ax_2^*$ term and $x_2^* = (1+x_1^*)(b-x_1^*)$ to eliminate the other x_2^*. Setting $\mathrm{Tr}[J(x_1^*, x_2^*)] = 0$ imposes a condition on x_1^* and b.

$$\begin{aligned}
0 &= b - \frac{(1+x_1^*)(b-x_1^*)}{(1+x_1^*)^2} - 2x_1^* + \frac{x_1^*}{1+x_1^*} - 2\frac{x_1^*}{1+x_1^*} \\
0 &= b - \frac{b-x_1^*}{1+x_1^*} - 2x_1^* - \frac{x_1^*}{1+x_1^*} \\
0 &= b - \frac{b}{1+x_1^*} - 2x_1^* \\
b &= (b-2x_1^*)(1+x_1^*) \\
b &= b + x_1^*(b-2) - 2(x_1^*)^2 \\
0 &= x_1^*\frac{b-2}{2} - (x_1^*)^2
\end{aligned}$$

Since $x_1^* \neq 0$, we have

$$x_1^* = \frac{b-2}{2}$$

Now, x_1^* can be eliminated by evaluating the intersection of the nullclines

$$\begin{aligned}
(1+x_1^*)(b-x_1^*) &= \frac{x_1^*}{a(1+x_1^*)} \\
\left(1+\frac{b-2}{2}\right)\left(b-\frac{b-2}{2}\right) &= \frac{\frac{b-2}{2}}{a(1+\frac{b-2}{2})} \\
\frac{b}{2}\frac{b+2}{2} &= \frac{\frac{b-2}{2}}{a\frac{b}{2}} \\
a &= \frac{4(b-2)}{(b+2)b^2}
\end{aligned}$$

Therefore, for a fixed b, the critical value $a_c(b)$ at which the Hopf bifurcation can occur is $a_c = \frac{4(b-2)}{(b+2)b^2}$. Since $a > 0$ then we must have $b > 2$ so that the bifurcation may exist.

(e) Check that the determinant of the linearised system is positive at that critical value. Together with the previous result, this implies that the eigenvalues are complex conjugate with zero real part supporting the hypothesis that a Hopf bifurcation is present.

The determinant is

$$\mathrm{Det}[J(x_1^*, x_2^*)] = \left(b - 2x_1^* - \frac{x_2^*}{(1+x_1^*)^2}\right)\left(\frac{x_1^*}{1+x_1^*} - 2ax_2^*\right) - \frac{x_2^*}{(1+x_1^*)^2}\frac{-x_1^*}{1+x_1^*}$$

Similarly use $x_2^* = \frac{x_1^*}{a(1+x_1^*)}$ to eliminate the $-2ax_2^*$ term and $x_2^* = (1+x_1^*)(b-x_1^*)$ to eliminate the other x_2^*. Also use $x_1^* = \frac{b-2}{2}$ to replace b with $2x_1^* + 2$.

$$\begin{aligned}
\mathrm{Det}[J(x_1^*, x_2^*)] &= \left((2x_1^*+2) - 2x_1^* - \frac{b-x_1^*}{1+x_1^*}\right)\left(\frac{x_1^*}{1+x_1^*} - 2\frac{x_1^*}{1+x_1^*}\right) - \frac{b-x_1^*}{1+x_1^*}\frac{-x_1^*}{1+x_1^*} \\
&= \left(2 - \frac{(2x_1^*+2)-x_1^*}{1+x_1^*}\right)\left(\frac{-x_1^*}{1+x_1^*}\right) - \frac{(2x_1^*+2)-x_1^*}{1+x_1^*}\frac{-x_1^*}{1+x_1^*} \\
&= \left(2 - \frac{2+x_1^*}{1+x_1^*}\right)\left(\frac{-x_1^*}{1+x_1^*}\right) + \frac{2+x_1^*}{1+x_1^*}\frac{x_1^*}{1+x_1^*} \\
&= \frac{2(1+x_1^*)-(2+x_1^*)}{1+x_1^*}\frac{-x_1^*}{1+x_1^*} + \frac{2+x_1^*}{1+x_1^*}\frac{x_1^*}{1+x_1^*} \\
&= \frac{x_1^*(-x_1^*)}{(1+x_1^*)^2} + \frac{2x_1^*+(x_1^*)^2}{(1+x_1^*)^2} \\
&= \frac{2x_1^*}{(1+x_1^*)^2} > 0
\end{aligned}$$

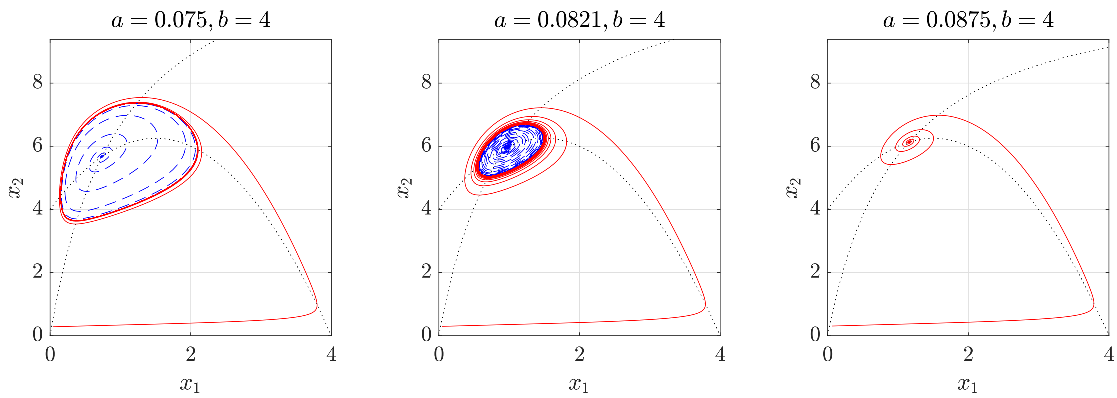
(f) Visualise the bifurcation using Matlab. Use a fixed $b = 4$, then vary the value of a from $0.95a_c(b)$ to $1.01a_c(b)$. At each value of a, plot the nullclines, numerically find the fixed point, and plot two trajectories, one starting close to the fixed point, the other starting close to the origin. Is this Hopf bifurcation super or sub-critical ?
Help : To find the roots of a polynomial equation in Matlab use the `roots` command.
To use the `roots` function we need the coefficients of the third order polynomial that derives from the intersection

$$(1+x_1)(b-x_1) = \frac{x_1}{a(1+x_1)}$$

that is

$$x_1^3 + x_1^2(2-b) + x_1(1+1/a-2b) - b = 0$$

See the code for the rest. The plots show that there is a limit cycle that closes into the unstable spiral fixed point, transforming it to a stable spiral. The bifurcation is supercritical.



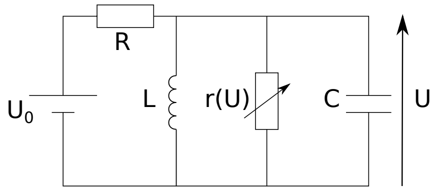
3 Van der Pol Oscillators

In the first half of the twentieth century, important research was done of the nonlinear dynamics of oscillators, motivated by the developpement of the radio and circuits with vacuum tubes. Many of these circuits have their dynamics governed by the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. For this equation, the Liénard theorem states that if

- $f(x)$ and $g(x)$ are continuously differentiable $\forall x$
- $g(x)$ is an odd function
- $g(x) > 0$ for $x > 0$
- $f(x)$ is an even function

· $F(x) = \int_0^x f(u)\mathrm{d}u$ is odd and vanishes at $x = a > 0$
· For $0 < x < a$, it is $F(x) < 0$ while for $x > a$, $F(x) > 0$ and is non decreasing.
· $\lim_{x\to\infty} F(x) = \infty$
then there is a unique stable limit cycle in the phase plane that surrounds the origin. One can understand this result qualitatively. In fact $-g(x)$ represents a nonlinear restoring force and $-f(x)\dot{x}$ a nonlinear damping force.

(a) A famous case of the Liénard equation is the Van der Pol equation. It can be derived from a circuit with a resistance, an inductance, a capacitor and a triode, which has a variable resistivity $r(U) \propto 1/U^2$, U being the applied voltage. The dynamics of the circuit is determined by the non-dimensionalised equation $\ddot{x}+\mu(x^2-1)\dot{x}+x = 0$, with $\mu > 0$, i.e. the Van der Pol equation. Show that Liénard's theorem applies to this equation.



The Van der Pol equation is the Liénard equation having identified $f(x) = \mu(x^2 - 1)$ and $g(x) = x$. The Liénard theorem does apply since :
· f, g are continuously differentiable.
· $g(-x) = -x = -g(x)$ is odd.
· $g(x) = x > 0$ when $x > 0$.
· $f(-x) = \mu((-x)^2 - 1) = \mu(x^2 - 1) = f(x)$ is even.
· $F(x) = \mu(\frac{1}{3}u^3 - u)|_{x=0}^{x} = \mu(\frac{1}{3}x^3 - x) = \frac{\mu}{3}x(x-\sqrt{3})(x+\sqrt{3})$. Since $\mu > 0$, the zeros are 0 and $\pm\sqrt{3}$, so only one positive zero. On the interval $0 < x < \sqrt{3}$, $F(x) < 0$, and $F(x) > 0$ non-decreasing for $x > \sqrt{3}$, and diverges as $x \to \infty$.

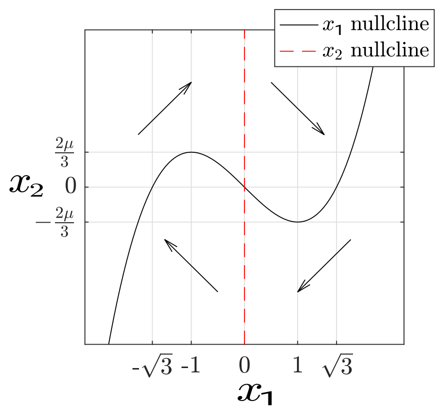
Therefore Liénard's theorem does apply, meaning there exists a unique limit cycle surrounding the origin.

(b) Prove that the differential equation system

$$\begin{cases} \dot{x}_1 = x_2 - \mu(\frac{1}{3}x_1^3 - x_1) \\ \dot{x}_2 = -x_1 \end{cases}$$

is equivalent to the Van der Pol equation. Qualitatively draw the phase portrait.

Differentiate the first equation to get $\ddot{x}_1 = \dot{x}_2 - \mu(x_1^2\dot{x}_1 - \dot{x}_1)$. Plug in the second equation to get $\ddot{x}_1 = -x_1 - \mu(x_1^2\dot{x}_1 - \dot{x}_1)$. The equation is indeed equivalent to the Van der Pol equation $\ddot{x}_1 + \mu(x_1^2 - 1)\dot{x}_1 + \dot{x}_1 = 0$, with $x = x_1$. To draw the phase a qualitative phase portrait, compute the nullclines. The x_1-nullcline is the polynomial $x_2 = \mu(\frac{1}{3}x_1^3 - x_1) = \frac{\mu}{3}x_1(x_1 - \sqrt{3})(x_1 + \sqrt{3})$. The x_2-nullcline is the vertical line $x_1 = 0$.



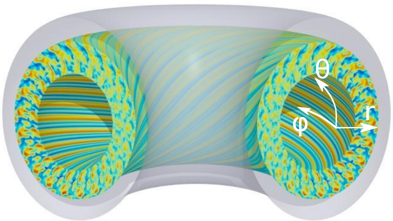
(c) Suppose that $\mu \ll 1$. In this case the limit cycle, C, is approximately a circle centered at the origin. Use the divergence theorem $\oint_C \dot{\vec{x}} \cdot \vec{n}\, \mathrm{d}l = \iint_A \vec{\nabla} \cdot \dot{\vec{x}}\, \mathrm{d}A$ to get the radius.

The left hand side of the divergence theorem is zero because along the limit cycle, $\dot{\vec{x}}$ is tangent to the cycle C, therefore perpendicular to the normal vector $\vec{n}$. For the right hand side $\vec{\nabla} \cdot \dot{\vec{x}} = \frac{\partial \dot{x}_1}{\partial x_1} + \frac{\partial \dot{x}_2}{\partial x_2} = -\mu(x_1^2 - 1)$. The integral is

$$\begin{aligned}
\iint_A \vec{\nabla} \cdot \dot{\vec{x}}\, \mathrm{d}A &= -\mu \int_0^R \int_0^{2\pi} (r^2 \cos^2\theta - 1) r \mathrm{d}\theta \mathrm{d}r \\
&= -\mu \int_0^R \int_0^{2\pi} \left(\frac{r^2}{2}(1 + \cos(2\theta)) - 1 \right) r \mathrm{d}\theta \mathrm{d}r \\
&= -\mu \int_0^R \left(\frac{r^2}{2}(2\pi + 0) - 2\pi \right) r \mathrm{d}r \\
&= -\mu\pi \left(\frac{R^4}{4} - R^2 \right) \\
&= -\frac{\mu\pi}{4} R^2 (R-2)(R+2)
\end{aligned}$$

Equating this to zero means that the limit cycle has a radius $R = 2$. The $R = 0$ corresponds to the origin, which is a fixed point.

4 L-H transition in tokamak plasmas



The mechanism underlying the transition from Low (L) to High (H) confinement mode in a tokamak plasma is still not yet fully understood. Generally, the improvement in confinement is associated to a locally reduced amplitude of turbulent fluctuations, which occurs when the power injected into the plasma exceeds a certain threshold that depends on the characteristics of the tokamak. The tokamak geometry is shown in the figure. The results of a numerical simulation of the turbulence in a tokamak is shown and the colors represent the amplitude of plasma density fluctuations. We call r, θ and φ the radial, poloidal and toroidal directions, respectively. Many different approaches have been attempted to describe the L-H transition.

The aim of this exercise is to study a model composed of two equations that describe plasma turbulence and its saturation, as proposed in *P. H. Diamond et al., Phys. Rev. Lett. 1994*. The first equation of the model can be written in the form :

$$\frac{1}{2}\frac{\partial \bar{E}}{\partial t} = \gamma_0 \bar{E} - a_1 \bar{E}^2 - a_2 \bar{E}\bar{U}$$

This equation represents the evolution of the energy associated with the turbulent fluctuations. Here $\bar{E} \equiv \left|\frac{\tilde{n}_k}{n_0}\right|^2$, where $\tilde{n}_k$ stands for the amplitude (standard deviation) of plasma density fluctuations, and n_0 the background density value. The terms $\gamma_0\bar{E}$ and $-a_1\bar{E}^2$ represent the linear growth of the instability driving the turbulence fluctuations and the turbulence saturation due to nonlinear stabilizing terms, respectively. The last term, $-a_2\bar{E}\bar{U}$, describes the turbulence saturation due to the shearing of the turbulent eddies caused by the poloidal flow, being $\bar{U} \equiv \left|\frac{\partial\langle V_\theta\rangle}{\partial r}\right|^2$ the energy associated with poloidal flows, V_θ the poloidal velocity, and $\langle\cdot\rangle$ operator representing the average over θ.

(a) We now derive the second equation of the model to state the evolution of $\bar{U}$:

$$\frac{1}{2}\frac{\partial \bar{U}}{\partial t} = -\mu\bar{U} + a_3\bar{E}\bar{U}$$

where $-\mu\bar{U}$ is the viscous damping of the poloidal flow and $a_3\bar{E}\bar{U}$ is the drive of the poloidal flow due to turbulence (by means of the Reynold stress mechanism). Derive this equation starting from the conservation of poloidal momentum :

$$\frac{\mathrm{d}V_\theta}{\mathrm{d}t} = \frac{\partial V_\theta}{\partial t} + \left(V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial\theta}\right)V_\theta = -\frac{1}{r\rho_0}\frac{\partial p}{\partial\theta} - \mu V_\theta$$

where the pressure gradient and the viscous damping of the poloidal flow are taken into account (μ is constant). The quantity ρ_0 is the background plasma density.
Average the poloidal momentum conservation over the θ direction, keeping in mind that the domain is periodic in θ. Separate velocities in a part averaged on θ and in a fluctuating one :

$$V_\theta(t,r,\theta) = \langle V_\theta\rangle(t,r) + \tilde{V}_\theta(t,r,\theta) \quad , \quad V_r\,(t,r,\theta) = \langle V_r\rangle(t,r) + \tilde{V}_r(t,r,\theta) \approx \tilde{V}_r(t,r,\theta)$$

Furthermore, the first approximation you need to consider is :

$$\left|\left\langle \tilde{V}_\theta\frac{\partial}{\partial r}\tilde{V}_r\right\rangle\right| \ll \left|\frac{\partial}{\partial r}\left[\langle\tilde{V}_r\tilde{V}_\theta\rangle\right]\right|.$$

and use the assumption :

$$\frac{\partial^2}{\partial r^2}\left[\langle\tilde{V}_r\tilde{V}_\theta\rangle\right] = -a_3\bar{E}\frac{\partial\langle V_\theta\rangle}{\partial r},$$

The term $\langle\tilde{V}_r\tilde{V}_\theta\rangle$ is the "Reynold stress". Due to this term small-scale turbulence fluctuations can drive large-scale flows in the poloidal direction.

The poloidal average of the right-hand side of the equation is

$$\Big\langle -\frac{1}{r\rho_0}\frac{\partial p}{\partial\theta} - \mu\big(\langle V_\theta\rangle + \tilde{V}_\theta\big)\Big\rangle = -\frac{1}{r\rho_0}\Big\langle\frac{\partial p}{\partial\theta}\Big\rangle - \mu\langle V_\theta\rangle = -\mu\langle V_\theta\rangle$$

where $\langle\frac{\partial p}{\partial\theta}\rangle = 0$ because of the periodicity in θ.
In addition, on the left-hand side, one has :

$$\Big\langle V_\theta\frac{1}{r}\frac{\partial}{\partial\theta}V_\theta\Big\rangle = \frac{1}{2r}\Big\langle\frac{\partial}{\partial\theta}V_\theta^2\Big\rangle = 0$$

that vanishes because of the periodicity in θ and

$$\Big\langle\tilde{V}_r\frac{\partial}{\partial r}\left[\langle V_\theta\rangle + \tilde{V}_\theta\right]\Big\rangle = \langle\tilde{V}_r\rangle\frac{\partial}{\partial r}\langle V_\theta\rangle + \Big\langle\frac{\partial}{\partial r}\left[\tilde{V}_r\tilde{V}_\theta\right]\Big\rangle - \Big\langle\tilde{V}_\theta\frac{\partial}{\partial r}\tilde{V}_r\Big\rangle =$$
$$\Big\langle\frac{\partial}{\partial r}\left[\tilde{V}_r\tilde{V}_\theta\right]\Big\rangle - \Big\langle\tilde{V}_\theta\frac{\partial}{\partial r}\tilde{V}_r\Big\rangle \approx \Big\langle\frac{\partial}{\partial r}\left[\tilde{V}_r\tilde{V}_\theta\right]\Big\rangle$$

where the approximation suggested in the statement of the problem has been used.
The poloidal average of the time derivative is

$$\Big\langle\frac{\partial V_\theta}{\partial t}\Big\rangle = \Big\langle\frac{\partial}{\partial t}\big(\langle V_\theta\rangle + \tilde{V}_\theta\big)\Big\rangle = \frac{\partial\langle V_\theta\rangle}{\partial t}$$

Therefore the poloidal averaged equation is :

$$\frac{\partial\langle V_\theta\rangle}{\partial t} = -\frac{\partial}{\partial r}\left[\langle\tilde{V}_r\tilde{V}_\theta\rangle\right] - \mu\langle V_\theta\rangle. \tag{1}$$

In order to complete the derivation of the second equation of the model, it is sufficient to apply the partial radial derivative to equation (1) and substitute the Reynolds stress with the given approximation. The final equation is simply retrived by multiplying both sides by $\partial\langle V_\theta\rangle/\partial r$.

The set of coefficients in the model, $(\gamma_0, a_1, a_2, a_3, \mu) \geq 0$, depends on the type of instability in the plasma, and on the considered wavelength of the fluctuations.
We perform here an analysis of the system. Despite its simplicity, the model shows stationary solutions corresponding to the L-mode, where the fluctuation level is high and the poloidal flow is low, and to the H-mode, where the poloidal flow limits the amplitude of fluctuations.

(b) Choosing $E = a_1\bar{E}/\gamma_0$, $U = a_2\bar{U}/\gamma_0$ and $\tau = t\gamma_0$, reduce the system to two equations dependent only on the parameters $a = a_3/a_1$ and $b = \mu/\gamma_0$.

In order to reduce the number of parameters, it is sufficient to multiply the first equation by a_1/γ_0^2 and the second equation by a_2/γ_0^2. The resulting system is :

$$\begin{cases} \frac{1}{2}\frac{\partial E}{\partial \tau} = E - E^2 - EU \\ \frac{1}{2}\frac{\partial U}{\partial \tau} = -bU + aEU \end{cases}$$

(c) Find the equilibrium points of the system. Distinguish the case $b > 0$ and $b = 0$. Discuss for each equilibrium point its physical meaning.

For $b > 0$, there are three equilibrium points :

$$(E = 0, U = 0) \quad , \quad (E = 1, U = 0) \quad , \quad (E = b/a, U = 1 - b/a)$$

The second equilibrium point corresponds to the L-mode. In fact, the energy associated with the poloidal flow gradient is null, and the turbulence is limited only by the diffusive saturation mechanism. The third equilibrium point is physically meaningful only in the case $a \geq b$ (i.e. $U \geq 0$) and it can be seen as the H-mode solution. In this regime, the poloidal flow gradient grows, and provides a factor of reduction of turbulent energy.

In case of $b = 0$, the equilibrium points reduce to the trivial solution, the point $(E = 1, U = 0)$, and to the $E = 0$ axis.

(d) Determine the nature (stability properties) of the equilibrium points. Consider b as a fixed parameter and vary a. Do you see a bifurcation ? Which bifurcation is it ? Discuss the physical meaning of this bifurcation and find the a value for which the L-H transition occurs.

The Jacobian matrix is :

$$\begin{pmatrix} 1 - 2E - U & -E \\ aU & aE - b \end{pmatrix}$$

The eigenvalues corresponding to the three equilibrium points are :

$$\lambda_1 = 1, \lambda_2 = -b \quad , \quad \lambda_1 = -1, \lambda_2 = a - b \quad , \quad \lambda_{1,2} = -\frac{b}{2a}\left(1 \pm \sqrt{1 + 4a\left(1 - \frac{a}{b}\right)}\right)$$

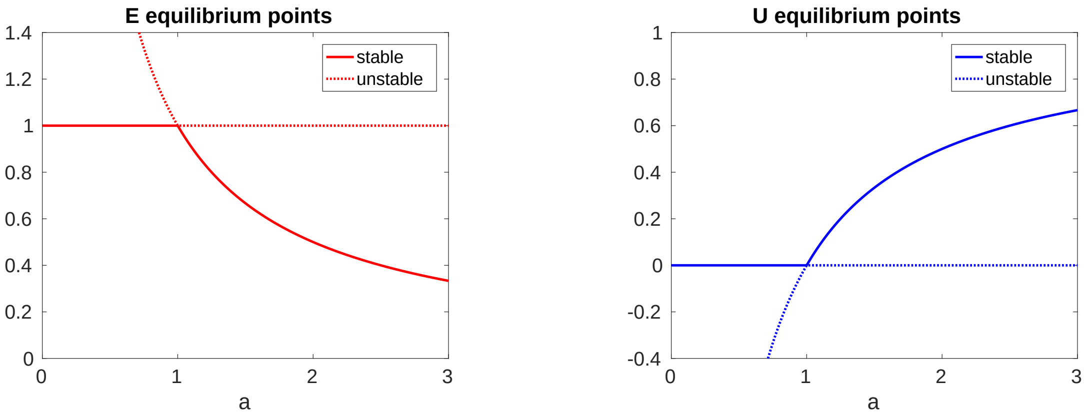
Therefore, the first equilibrium point is saddle node and is always unstable, meaning that for initial conditions different from $(E_0 = 0, U_0 = 0)$, turbulence and/or poloidal flows develop in the system. The second equilibrium point is stable for $a < b$. The third equilibrium point is always stable (when it exists, i.e. for $a \geq b$). We note also that the eigenvalues for this equilibrium point have a finite immaginary part for $a > b(1 + \sqrt{1 + 1/b})/2$. In this case, the convergence to the equilibrium point is characterized by an oscillatory behavior.

The L-H transition corrisponds to a transcritical bifurcation and it occurs when $a = b$.

We note that for $b = 0$, the only stable equilibrium points are $(E = 0, U > 1)$.

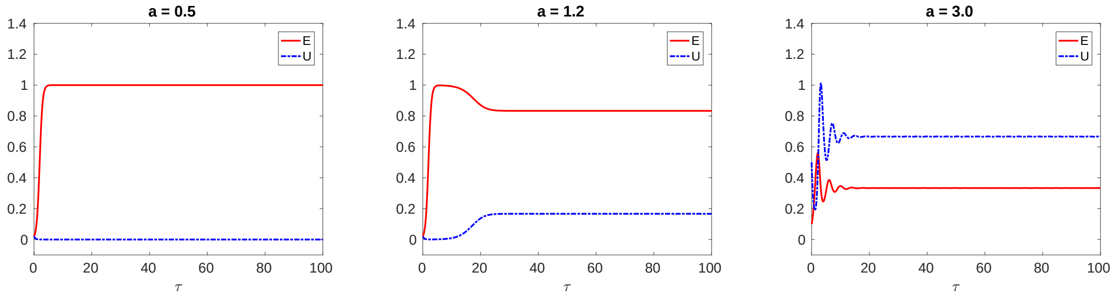
(e) Trace a bifurcation diagram of E and U as a function of the control parameter a, considering $b = 1$.

The bifurcation diagrams for the stationary points of E and U are shown in the following figures.



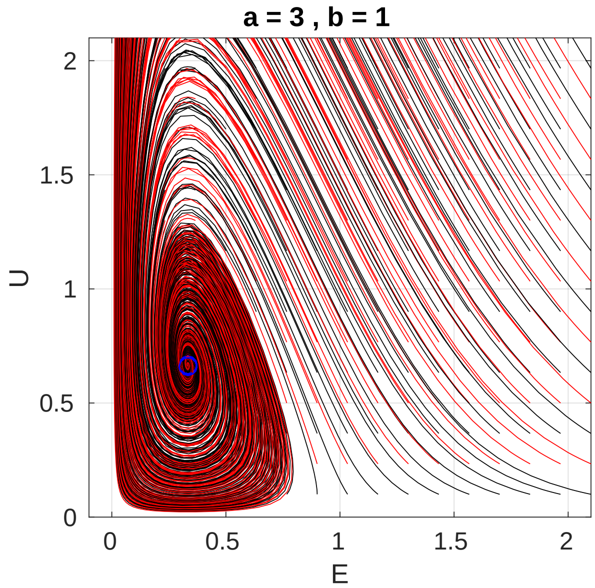
(f) Integrate numerically the system, for different values of the control parameter a, above and below the bifurcation found in point (d). Consider $b = 1$. Verify that in the case of eigenvalues with an imaginary part, the convergence to the equilibrium point is oscillatory.

The matlab script `L_H_transition.m` solves the given system of equations using the `ode45` function. The figures show the temporal evolution of the system, corresponding to $a = 0.5$ (convergence to L-mode), $a = 1.2$ (smooth convergence to H-mode) and $a = 3$ (oscillatory convergence to H-mode).



(g) Trace the phase space diagram in the particular case $a = 3$ and $b = 1$, and in the domain $0.1 < E < 2.1$ and $0.1 < U < 2.1$. Is there a globally stable equilibrium point ?

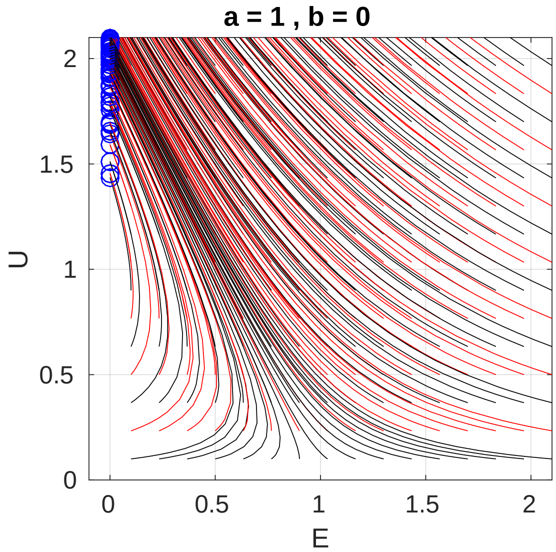
The last part of the script `L_H_transition.m` plots the trajectories in the phase space. A circle underlines the end point of each trajectory.



One can see from the figure that the system has a globally stable equilibrium point.

(h) Consider the case $a = 1$ and $b = 0$. In this case, the dynamics of the system includes no damping of the poloidal flows. Trace the phase space of the dynamical system in the domain $0.1 < E < 2.1$ and $0.1 < U < 2.1$. Comment on the behaviour of the system.

The figure shows the phase space in the case $a = 1$ and $b = 0$.



One can notice from the figure that trajectories converge to stable points on the $E = 0$ axis. Moreover, only solutions with $U > 1$ are stable. Therefore, turbulence energy introduced in the system through the initial conditions, is transferred progressively to poloidal flows, up to the point when the turbulence energy is completely exhausted. There is no other mechanism limiting the growth of poloidal flows. The final amplitude of the poloidal flow is dependent only on the initial conditions imposed to the system.