25 March 2025

Solutions 5 : Two-Dimensional Systems

1 Linearisation

Consider the two-dimensional system

$$\begin{cases} \dot{x}_1 = f_1 = \cos x_1 + \sin x_2 \\ \dot{x}_2 = f_2 = \cos x_1 \end{cases} \tag{1}$$

(a) Identify the x_2 nullclines.

We first solve $\dot{x}_2 = 0 = \cos x_1$. The solution is a set of vertical straight lines $\{x_1 = \pi(n + 1/2), n \in \mathbb{Z}\}$.

(b) Identify the x_1 nullclines.

We now solve $\dot{x}_1 = 0$, that is

$$0 = \cos x_1 + \sin x_2$$

This can also be written as

$$\sin(-x_2) = \sin(x_1 + \pi/2)$$

We note that $\sin\alpha = \sin\beta$ if $\alpha = \beta + 2\pi n$ or if $\alpha = \pi - \beta + 2\pi n$, $n \in \mathbb{Z}$. This means there are two infinite sets of solutions, the points that satisfy $-x_2 = (x_1 + \pi/2) + 2\pi n$, and those such that $-x_2 = \pi - (x_1 + \pi/2) + 2\pi n$. These are two infinite sets of parallel lines $\{x_2 = x_1 + \pi(2n - 1/2), n \in \mathbb{Z}\}$ and $\{x_2 = -x_1 + \pi(2n - 1/2), n \in \mathbb{Z}\}$.

(c) Find all the fixed points.

For system (1) the fixed points are the intersections of the x_1 and x_2 nullclines. A point is on a $\dot{x}_2 = 0$ nullcline, if $x_1 = \pi(n + 1/2), n \in \mathbb{Z}$. A point is on the x_1 nullcline, if $x_2 = \pi(n + 1/2) + \pi(2m - 1/2)$ with $n, m \in \mathbb{Z}$. This leads to the fixed points $\vec{x}^{\,*} = \big(\pi(n + 1/2), \pi(n + 2m)\big)$ with $n, m \in \mathbb{Z}$. The second possibility implies $x_2 = -\pi(n + 1/2) + \pi(2m - 1/2) = \pi(-n + 2m - 1)$ with $n, m \in \mathbb{Z}$. this leads to the fixed points $\vec{x}^{\,*} = \big(\pi(n + 1/2), \pi(1 - n + 2m)\big)$ with $n, m \in \mathbb{Z}$.

(d) Linearise the system around the equilibrium points and, if possible, identify their nature (stable, unstable, center, stable spiral, unstable spiral ...).

The functions f_1 and f_2 are 2π-periodic in both x_1 and x_2. Thus a translation in $(2\pi n, 2\pi m)$ with $n, m \in \mathbb{Z}$ leaves the functions unchanged. Therefore, a square of side 2π and the fixed points inside (or possible on the adjacent boundaries) can be considered representatives of all fixed points. We choose the square $[-\pi, \pi] \times [-\frac{\pi}{2}, \frac{3\pi}{2}]$. Inside this square, the fixed points are $(\frac{\pi}{2}, 0)$, $(\frac{\pi}{2}, \pi)$, $(-\frac{\pi}{2}, 0)$ and $(-\frac{\pi}{2}, \pi)$ (and there are no fixed points on the boundaries). All other fixed points are translations by $(2\pi n, 2\pi m)$ of these four points.

Near $(\frac{\pi}{2}, 0)$ we have

$$\begin{cases} \dot{x}_1 = \cos(\frac{\pi}{2} + \delta x_1) + \sin(0 + \delta x_2) \approx -\delta x_1 + \delta x_2 \\ \dot{x}_2 = \cos(\frac{\pi}{2} + \delta x_1) \approx -\delta x_1 \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of the matrix are $\frac{-1 \pm i\sqrt{3}}{2}$. This is a stable spiral.

Near $(\frac{\pi}{2}, \pi)$ we have

$$\begin{cases} \dot{x}_1 = \cos(\frac{\pi}{2} + \delta x_1) + \sin(\pi + \delta x_2) \approx -\delta x_1 - \delta x_2 \\ \dot{x}_2 = \cos(\frac{\pi}{2} + \delta x_1) \approx -\delta x_1 \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of the matrix are $\frac{-1-\sqrt{5}}{2} < 0$ and $\frac{-1+\sqrt{5}}{2} > 0$. This is a saddle point.
Near $(-\frac{\pi}{2}, 0)$ we have

$$\begin{cases} \dot{x}_1 = \cos(-\frac{\pi}{2} + \delta x_1) + \sin(0 + \delta x_2) \approx \delta x_1 + \delta x_2 \\ \dot{x}_2 = \cos(-\frac{\pi}{2} + \delta x_1) \approx \delta x_1 \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

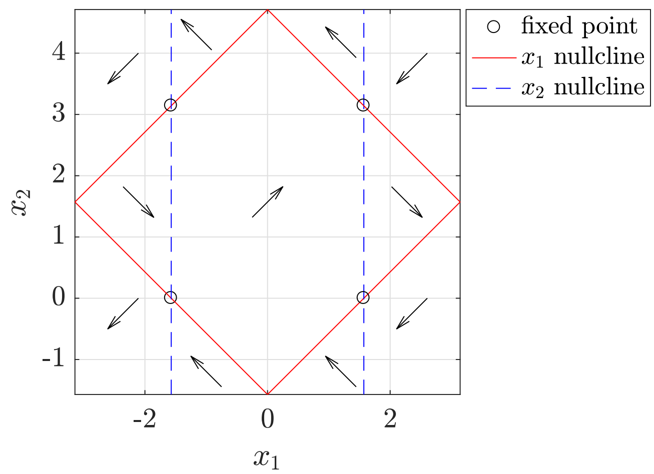
The eigenvalues of the matrix are $\frac{1-\sqrt{5}}{2} < 0$ and $\frac{1+\sqrt{5}}{2} > 0$. This is a saddle point.
Near $(-\frac{\pi}{2}, \pi)$ we have

$$\begin{cases} \dot{x}_1 = \cos(-\frac{\pi}{2} + \delta x_1) + \sin(\pi + \delta x_2) \approx \delta x_1 - \delta x_2 \\ \dot{x}_2 = \cos(-\frac{\pi}{2} + \delta x_1) \approx \delta x_1 \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of the matrix are $\frac{1 \pm i\sqrt{3}}{2}$, so this is an unstable spiral.

(e) Draw qualitatively the phase portrait by indicating the nullclines and arrow direction in each sector.

For system (1), the plot is



2 Linearisation II

Repeat the first exercise with the following system :

$$\begin{cases} \dot{x}_1 = x_1 x_2^2 - x_1^2 \\ \dot{x}_2 = x_2(x_2 - 1) \end{cases} \tag{2}$$

We solve $\dot{x}_2 = 0 = x_2(x_2 - 1)$. The solutions are the horizontal lines $x_2 = 0$ and $x_2 = 1$.
We solve $0 = x_1 x_2^2 - x_1^2 = x_1(x_2^2 - x_1)$ which means that $x_1 = 0$ or $x_1 = x_2^2$.
The fixed points are the intersections of the x_1 and x_2 nullclines. If $x_2 = 0$, the first possibility is that $x_1 = 0$ giving the fixed point $(0, 0)$. The second possibility that $x_1 = x_2^2 = 0^2$ gives the same point. If $x_2 = 1$, the first possibility is that $x_1 = 0$ giving the fixed point $(0, 1)$. The second possibility that $x_1 = x_2^2 = 1^2$ gives the point $(1, 1)$.
Near $(0, 0)$ we have

$$\begin{cases} \dot{x}_1 = \delta x_1 \delta x_2^2 - \delta x_1^2 = 0 + \mathcal{O}(||\delta \vec{x}||_2^2) \\ \dot{x}_2 = \delta x_2(\delta x_2 - 1) = -\delta x_2 + \mathcal{O}(||\delta \vec{x}||_2^2) \end{cases} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \approx \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The eigenvalues of the matrix are 0 and -1. Because of the null eigenvalue, this is a marginal case and non-linear terms matter. A more detailed study would be necessary to evaluate the nature of this fixed point.
Near $(0,1)$ we have

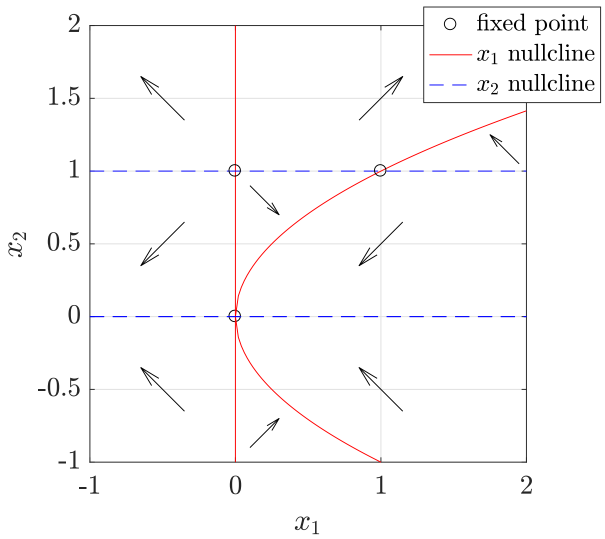
$$\begin{cases} \dot{x}_1 = \delta x_1(1+\delta x_2)^2 - \delta x_1^2 = \delta x_1[1+\mathcal{O}(||\delta\vec{x}||_2)] \\ \dot{x}_2 = (1+\delta x_2)(1+\delta x_2 - 1) = \delta x_2 + \mathcal{O}(||\delta\vec{x}||_2^2) \end{cases} \Leftrightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} \approx \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The eigenvalue of the matrix is 1, with a geometric multiplicity of 2. This is a repeller.
Near $(1,1)$ we have

$$\begin{cases} \dot{x}_1 = (1+\delta x_1)(1+\delta x_2)^2 - (1+\delta x_1)^2 \approx (1+\delta x_1+2\delta x_2) - (1+2\delta x_1) \approx -\delta x_1 + 2\delta x_2 \\ \dot{x}_2 = (1+\delta x_2)(1+\delta x_2-1) \approx \delta x_2 \end{cases}$$
$$\Leftrightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} \approx \begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The eigenvalues of the matrix are 1 and -1. This is a saddle point.

To plot the arrow directions, we inspect at the sign of $\dot{x}_1$ and $\dot{x}_2$ in each sector. For example $\dot{x}_2$ is negative (so the arrow points down) if $x_2 \in]0,1[$, otherwise it is positive (the arrow points up). Also $\dot{x}_1$ is positive (arrow points left) for points in between the $x_1 = 0$ vertical line and the $x_1 = x_2^2$ rotated parabola. For other points the arrows go right.



3 Epidemic Model

Some of the first epidemiological models were conceived by Kermarck and McKendrick in the 1920s. Nowadays a great range of more complicated models exist (we will explore them later in the semester). Among the models developed by Kermarck and McKendrick we consider the SIR model. The letter S stands for susceptible (healthy individuals that can get infected), I for infected and R for individuals that have recovered from the illness (if we suppose that this is not a deadly disease). The model is expressed as

$$\begin{cases} \dot{S} = -\alpha SI \\ \dot{I} = \alpha SI - \beta I \\ \dot{R} = \beta I \end{cases}$$

The susceptible individuals get infected at a rate proportional to the encounters they have with infected individuals explaining the term αSI, with $\alpha > 0$. Also the rate at which people recover from the disease is proportional to the number of infected individuals, explaining βI, with $\beta > 0$. In this model people who recover from the disease become immune and cannot get infected again. In cases like smallpox this is a good model. Here it is assumed that the total population stays constant, i.e. births, deaths, and migrations occur on a time scale much slower than the spread of the disease.

(a) Show that indeed the total population is constant. Then, show that the dynamics of the R variable is determined by the two other variables, meaning that we can eliminate R, and reduce the SIR model to a two-dimensional model.

The total population is $T = S + I + R$, and its derivative is

$$\dot{T} = \dot{S} + \dot{I} + \dot{R} = -\alpha SI + \alpha SI - \beta I + \beta I = 0$$

This means the total population is constant $T(t) = T(0) = S_0 + I_0 + R_0$ and the R variable can be eliminated using $R(t) = T(t) - S(t) - I(t) = T(0) - S(t) - I(t)$. The system reduces therefore to

$$\begin{cases} \dot{S} = -\alpha SI \\ \dot{I} = \alpha SI - \beta I \end{cases}$$

(b) Find the fixed points of the system. Identify, when possible, the stable and unstable manifolds of the linearised system.

We solve $\dot{S} = 0 = -\alpha SI$, finding that $S = 0$ and $I = 0$ are the S nullclines. Then, we solve $\dot{I} = 0 = \alpha SI - \beta I$. This shows that $I = 0$ and $S = \beta/\alpha$ are the I nullclines. Therefore all points of the line $I = 0$ are fixed points. This makes sense : if $I = 0$, then the number of infected individuals remains constant, independently of S. To classify the fixed points, we first calculate the Jacobian matrix.

$$\begin{bmatrix} \frac{\partial \dot{S}}{\partial S} & \frac{\partial \dot{S}}{\partial I} \\ \frac{\partial \dot{I}}{\partial S} & \frac{\partial \dot{I}}{\partial I} \end{bmatrix} = \begin{bmatrix} -\alpha I & -\alpha S \\ \alpha I & \alpha S - \beta \end{bmatrix}$$

Near the fixed points $(S, 0)$ we have

$$\begin{bmatrix} \dot{S} \\ \dot{I} \end{bmatrix} \approx \begin{bmatrix} 0 & -\alpha S \\ 0 & \alpha S - \beta \end{bmatrix} \begin{bmatrix} S \\ I \end{bmatrix}$$

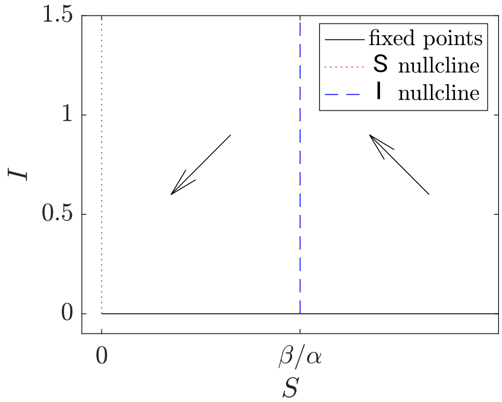
The eigenvalues are 0 and $\alpha S - \beta$. Because of the presence of a 0 eigenvalue, it is not possible to directly identify the nature of the fixed points. The eigenvectors are

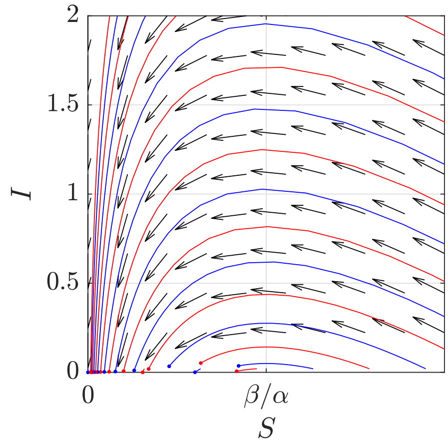
$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_{\alpha S - \beta} = \begin{bmatrix} 1 \\ \frac{\beta}{\alpha S} - 1 \end{bmatrix}$$

The $\vec{v}_{\alpha S-\beta}$ direction is an unstable manifold of the linearised system if $S > \beta/\alpha$, since it is associated to a positive eigenvalue in that case, and a stable manifold if $S < \beta/\alpha$. The zero eigenvalue is due to the non-isolated nature of the stationary points and the linearisation does not predict what happens in this direction.

(c) Plot qualitatively the phase portrait by indicating the nullclines and the arrow direction in each sector. Find the physical interpretation of this sketch. Complete the analysis with a numerical plot.

For the whole phase space $S, I > 0$ we have S is monotonically decreasing. For the I direction $\dot{I} < 0$ if $S < \beta/\alpha$ and $\dot{I} > 0$ if $S > \beta/\alpha$.





The sketch shows that the number of healthy susceptible people only decreases, i.e. a disease spreads. Also, if the number of susceptible people is sufficiently large (over β/α) the rate at which people get infected is bigger than the rate at which people recover meaning the number of infected people in the population increases. The numerically computed phase portrait shows more information. Trajectories starting with $S_0, I_0 > 0$ converge to a point $(S_f, 0)$ with $0 < S_f < \beta/\alpha$. This means that when a disease emerges in a population, a non zero number of susceptible people will never get infected (if the disease is deadly there will always be some survivors). However, the number of survivors cannot be greater than β/α.

(d) An epidemic occurs when the number of infected people increases initially. Find the initial conditions for an epidemic to occur. Discuss the physical meaning of varying the parameters α and β.

An epidemic occurs when $\dot{I}(t=0) = \alpha S_0 I_0 - \beta I_0 > 0$ which means $I_0 > 0$ and $S_0 > \beta/\alpha$. The first condition $I_0 > 0$ means that an infected person should exist in order to get an epidemic. There is a lower bound on the number of susceptible people $S_0 = \beta/\alpha$ to get an epidemic, a bound that decreases as α increases, meaning a more contagious disease needs a smaller group of susceptible people to become an epidemic. The limit increases as β increases. Indeed if people recover quickly, a larger group of susceptible people is needed for the disease to become an epidemic.

(e) Find analytically the trajectories $I = I(S)$ and compare them with the numerical solutions.

First we consider the case $I_0 = 0$. In this case the solution is stationary, with $I(t) = 0$ and $S(t) = S_0$. Second, if $S_0 = 0$ the solution is $S(t) = 0$ (since $\dot{S} = 0$) and the trajectory $I(t) = I_0 e^{-\beta t}$ (since $\dot{I} = -\beta I$). Otherwise if $I, S \neq 0$, the trajectory can be found using

$$\frac{\dot{I}}{\dot{S}} = \frac{\alpha S I - \beta I}{-\alpha S I}$$

that is

$$\frac{\mathrm{d}I}{\mathrm{d}S} = \frac{\beta}{\alpha S} - 1$$

We can integrate

$$\int_{I_0}^{I(t)} \mathrm{d}I = \int_{S_0}^{S(t)} \left(\frac{\beta}{\alpha S} - 1\right) \mathrm{d}S$$

to obtain

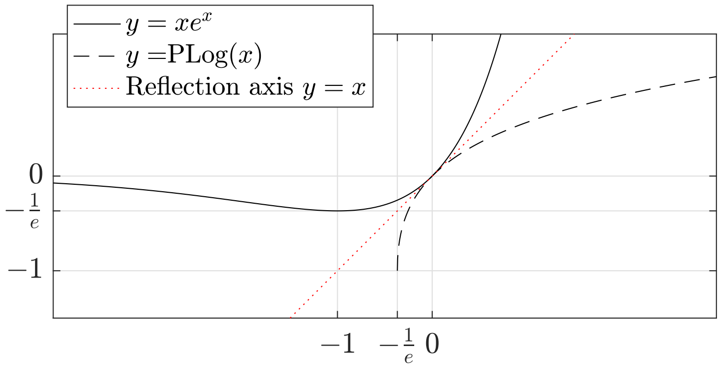
$$I(S) = -S + \frac{\beta}{\alpha} \ln(S/S_0) + S_0 + I_0$$

(f) Find and discuss the limit (S_f, I_f) of the system as $t \to +\infty$.

For initial conditions that are fixed points ($I_0 = 0$), the solution is constant, $(S_f, I_f) = (S_0, 0)$. Otherwise if $S_0 = 0$, as derived in question (e), S remains constant while I decreases exponentially to zero, therefore $(S_f, I_f) = (0, 0)$. For other initial conditions $(S_0, I_0 > 0)$, the trajectory is $I(S) = S + \frac{\beta}{\alpha}\ln(S/S_0) - S_0 + I_0$. From the phase portrait one deduces that setting I to zero and solving for S gives two solutions, the smaller one is the value of S for $t \to \infty$ and the larger is the value of S for $t \to -\infty$ ($\dot{S} < 0$). Indeed

$$\begin{aligned}
0 &= -S_f + \frac{\beta}{\alpha}\ln(S_f/S_0) + S_0 + I_0 \\
-S_0 - I_0 + \frac{\beta}{\alpha}\ln S_0 &= -S_f + \frac{\beta}{\alpha}\ln S_f \\
\frac{-\alpha(S_0 + I_0)}{\beta} + \ln S_0 &= -\frac{\alpha S_f}{\beta} + \ln S_f \\
\frac{-\alpha}{\beta} e^{\frac{-\alpha(S_0+I_0)}{\beta}} S_0 &= \frac{-\alpha S_f}{\beta}\exp\frac{-\alpha S_f}{\beta} \\
A &= Be^B
\end{aligned}$$

For $-1/e < A < 0$ there are two real solutions to $A = Be^B$, the first one $B \in]-\infty, -1[$ and the second $B \in]-1, 0[$. Out of these two, the one corresponding to S_f is the largest one $B \in]-1, 0[$, because it corresponds to the smallest of the two S. We therefore use the product logarithm (also called the Lambert W function) $\mathrm{PLog} : [-1/e, \infty[\to [-1, \infty[$, which when given A, returns the value in $B \in [-1, \infty[$ such that $A = Be^B$.



$$\begin{aligned}
\mathrm{PLog}\left(\frac{-\alpha}{\beta} e^{\frac{-\alpha(S_0+I_0)}{\beta}} S_0\right) &= \frac{-\alpha S_f}{\beta} \\
\frac{-\beta}{\alpha}\mathrm{PLog}\left(\frac{-\alpha S_0}{\beta} e^{\frac{-\alpha(S_0+I_0)}{\beta}}\right) &= S_f
\end{aligned}$$

We can prove more rigorously that the trajectories converge to an $I = 0$ fixed point. We know that S monotonically decreases since $\dot{S} < 0$. As a consequence, it has to converge to an S nullcline. This is not $S = 0$ because the trajectory formula implies that I, which must stay positive, goes to $-\infty$ as S vanishes, because of the logarithmic term. Therefore the trajectory converges to a point on the $I = 0$ line.