18 March 2025

Solutions 4 : Bifurcations and Two-Dimensional Systems

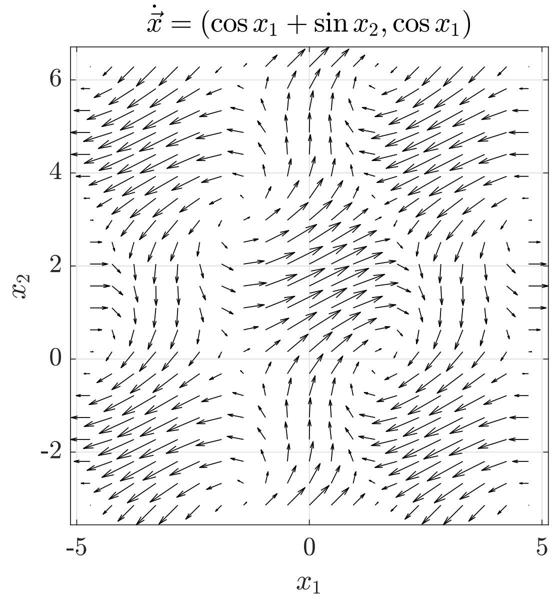
1 Numerical Phase Portrait

In this problem, we numerically produce the phase portrait of a two-dimensional system of differential equations using Matlab. Consider the differential equation

$$\begin{cases} \dot{x}_1 = \cos x_1 + \sin x_2 \\ \dot{x}_2 = \cos x_1 \end{cases}$$

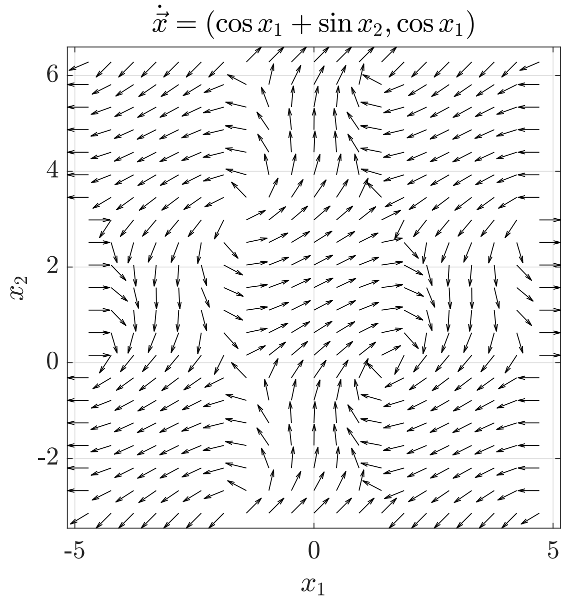
(a) Use Matlab's `meshgrid` and `quiver` functions to plot the vector field associated with the differential equation. Take the periodicity into consideration to determine the area on which to plot.

See the code for the solution. The command `[X1,X2] = meshgrid(x1,x2)` uses the two discretised vectors `x1` and `x2` to create the matrices `X1` and `X2`. They are defined as `X1(i,j)=x1(j)`, with only the x_1 coordinates and `X2(i,j)=x2(i)`, with only the x_2 coordinate. Finally, use the command `quiver(X1,X2,dX1,dX2)` that plots the vector field. The matrix elements `dX1(i,j)` contains $\dot{x}_1$ at `x1(j)` and `x2(i)`. Similarly `dX2(i,j)` contains $\dot{x}_2$ at `x1(j)` and `x2(i)`.



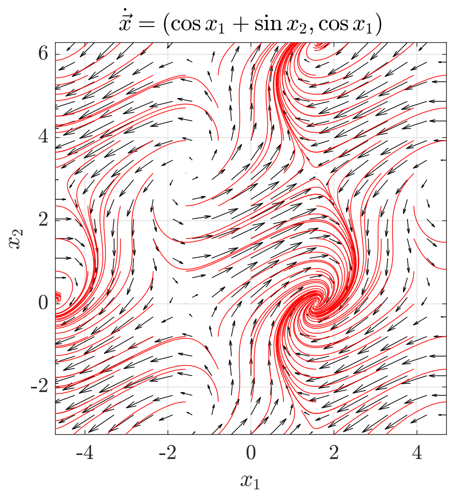
(b) Sometimes, due to strong variations in the vector field, some arrows are very small and hard to see. In some cases we are more interested in the direction of the vector rather than in the norm. Plot the vector field again, having renormalized the vectors to the unitary length.

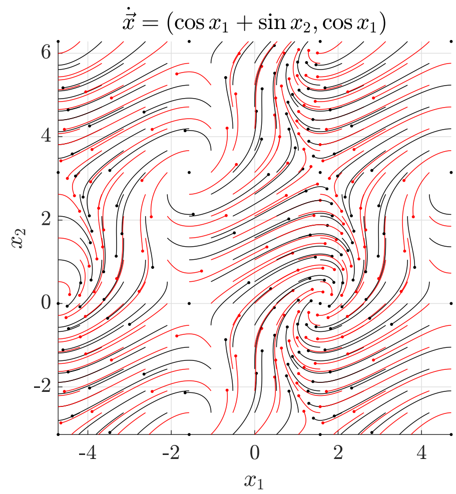
See the code for the solution. Use something like `norm_dX1 = dX1./sqrt(dX1.∧2+dX2.∧2)` to normalise the vector field. In Matlab, the dot '.' indicates that these are not the standard matrix operations but element by element operations. For example `B = A.∧2` means that `B(i,j)=A(i,j)∧2`, and not the usual element of the squared matrix.



(c) Another way to visualise a vector field is to plot the streamlines. Generate a grid of coordinates, which we define as the initial positions of the streamlines. For each of these initial conditions integrate the differential equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ up to a certain time t_f. Plot each trajectory. You can adjust t_f and refine or coarse the initial condition grid, add different symbols to the beginning and end of the trajectories to indicate the directions, alternate colors...

See the code for the solution.



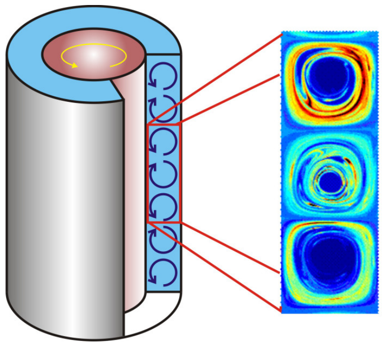


(d) Based on the numerical phase portrait you obtained, discuss qualitatively the behaviour of the solution (fixed points, their nature...).

The vector field is 2π-periodic, meaning all points separated by $(2\pi n, 2\pi m)$, with $n, m \in \mathbb{Z}$ are identical translations of each other. One only needs to analyse a square of side 2π. There are four different fixed points. At $(-\frac{\pi}{2}, 0)$ is a saddle point, at $(\frac{\pi}{2}, 0)$ there is a stable spiral, at $(-\frac{\pi}{2}, \pi)$ there is an unstable spiral and finally at $(\frac{\pi}{2}, \pi)$ there is another saddle point. All trajectories converge to a nearby stable spiral fixed points, except the ones starting on the stable manifold of the saddle points or on an unstable spiral fixed point.

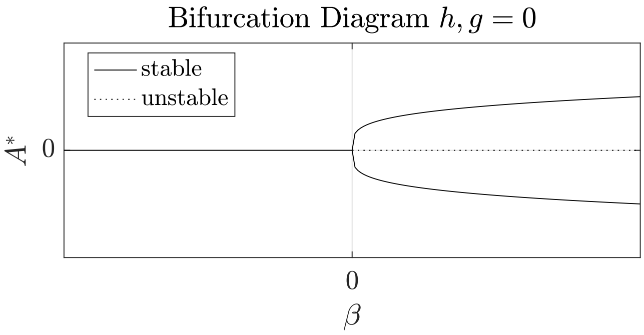
2 Bifurcation in a Fluid

The Taylor-Couette flow occurs when a fluid is placed in the gap between two co-axial cylinders rotating around their axis at different speeds. If the external cylinder is kept fixed and the inner one is rotating sufficiently slowly, a stable stationary flow is observed. Analytical calculations show that the azimuthal velocity of the flow is $v_\theta = C_1 r + C_2/r$. When the angular velocity of the inner cylinder exceeds a threshold, a smaller secondary flow appears on top of the fluid stationary flow. Its amplitude, $A(t)$, follows the simplified equation $\dot{A} = h + \epsilon A - gA^3 - kA^5$. The sign of A indicates the direction of the perturbed flow. The symmetry breaking parameter h represents possible imperfections of the system. The parameter k is always positive, preventing A from diverging. The system is said supercritical if $g > 0$ and subcritical if $g < 0$.



(a) First assume $g = 0$ and $h = 0$. Normalise the system so that it depends only on one parameter. Then plot qualitatively the bifurcation diagram for that parameter. Identify the type of bifurcation.

To redimensionalise we can divide the equation by k, $\frac{1}{k}\dot{A} = \frac{\epsilon}{k}A - A^5$, and define $\tau = kt$ and $\beta = \frac{\epsilon}{k}$. This leads to the equation $\frac{\mathrm{d}A}{\mathrm{d}\tau} = \beta A - A^5$. To find the fixed points, solve $0 = \beta A^* - A^{*5}$. $A^* = 0$ is always a solution. Other solutions appear for $0 = \beta - A^{*4}$, i.e. $A^* = \pm\sqrt[4]{\beta}$. These two additional points exist only if $\beta > 0$. To evaluate the stability of the fixed points, we compute the derivative $\frac{\partial}{\partial A}(\frac{\mathrm{d}A}{\mathrm{d}\tau}) = \beta - 5A^4$. At the origin this is $\left[\frac{\partial}{\partial A}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right]_{A=0} = \beta$ therefore the origin is stable for $\beta < 0$ and unstable for $\beta > 0$. For the other two points, $\left[\frac{\partial}{\partial A}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right]_{A=\pm\beta^{1/4}} = -4\beta$. These fixed points only exist when $\beta > 0$ where they are therefore always stable. We have a supercritical pitchfork diagram.



(b) To see the effect of an imperfection on our system, take $h > 0$ (still with $g = 0$). Find the critical value $\epsilon = \epsilon_c(h, k)$ at which the number of fixed points changes. Plot qualitatively the bifurcation diagram A^* in terms of ϵ. Discuss the limit of h going to zero.

For $\epsilon \leq 0$ the function $h + \epsilon A - kA^5$ is monotonic decreasing, so there is only one stable fixed point at some $A^* > 0$. When $\epsilon > 0$ is sufficiently large, more stable points appear. The critical case is when the local minimum touches the A axis. The local extrema are for $0 = \frac{\partial \dot{A}}{\partial A}$. Since $\frac{\partial \dot{A}}{\partial A} = \epsilon - 5kA^4$ the extrema are at $A_\pm = \pm\sqrt[4]{\frac{\epsilon}{5k}}$. With $h, \epsilon > 0$ we want to look at the value of $\dot{A}$ at the minimum located at A_-. The critical case is when

$$0 = h + \epsilon A_- - kA_-^5$$

Introducing the expression for A_- we have

$$0 = h - \epsilon(\frac{\epsilon}{5k})^{1/4} + k(\frac{\epsilon}{5k})^{5/4}$$

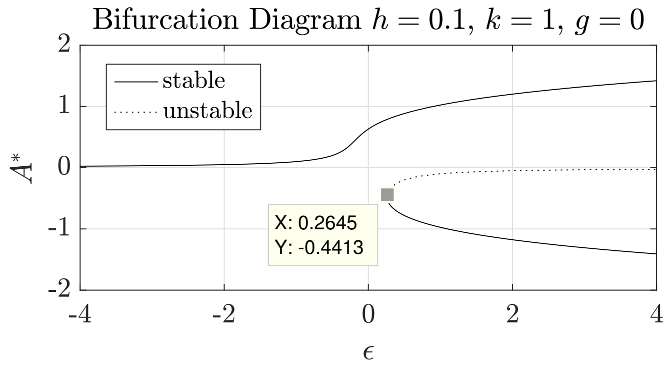
That is

$$\epsilon^{5/4}\frac{1}{(5k)^{1/4}}(1-\frac{1}{5}) = h$$

which leads to

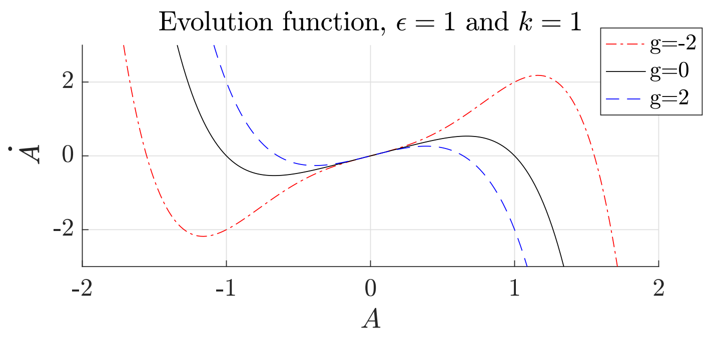
$$\epsilon_c(h,k) = \frac{5h^{4/5}k^{1/5}}{4^{4/5}}$$

Indeed as h goes to zero, so does ϵ_c, making the bifurcation diagram look more and more like the supercritical pitchfork. An example of bifurcation diagram for $k = 1$ and $h = 0.1$ is shown in the figure below. In this case $\epsilon_c(0.1, 1) = \frac{5\cdot 0.1^{4/5}}{4^{4/5}} \approx 0.26$.



The bifurcation diagram we observe can be interpreted as the result of a loss of symmetry. When $h = 0$, and ϵ reaches the splitting point of the pitchfork, the system can tend either to the positive or to the negative values of A^*. With $h > 0$ the system only tends to positive A^* when increasing ϵ.

(c) Neglect again the imperfections ($h = 0$). Using $\epsilon > 0$, find the fixed points. Then, discuss qualitatively their stability properties and show that the sign of g cannot change them.



There are three stationary points, one unstable at $A^* = 0$ and two stable symmetric around $A = 0$. To find them solve $0 = \epsilon A^* - gA^{*3} - kA^{*5}$. You can factor out $A^* = 0$ and then solve $0 = \epsilon - gA^{*2} - kA^{*4}$. Defining $A^{*2} = \alpha > 0$, we obtain a second order equation $\alpha^2 + \frac{g}{k}\alpha = \frac{\epsilon}{k}$, that is $(\alpha + \frac{g}{2k})^2 = \frac{4\epsilon k + g^2}{4k^2}$ and $\alpha_\pm = \frac{-g\pm\sqrt{4\epsilon k+g^2}}{2k}$. Both $\alpha_\pm$ exist since $\epsilon, k > 0$. Note that $\epsilon k > 0$ implies $4\epsilon k + g^2 > g^2$ and therefore $\sqrt{4\epsilon k + g^2} > |g|$. As a consequence, $\alpha_+ > 0$ and $\alpha_- < 0$. Only the positive α is an acceptable solution. Therefore besides the origin, the other fixed points are

$$A^* = \pm\sqrt{\alpha_+} = \pm\sqrt{\frac{-g + \sqrt{4\epsilon k + g^2}}{2k}}$$

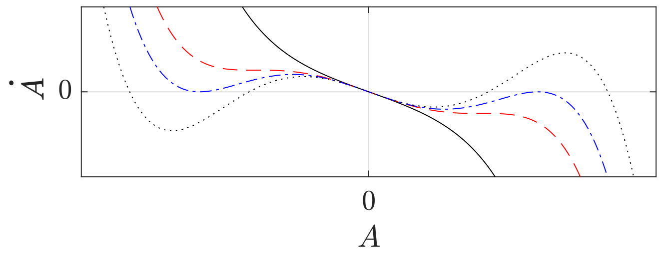
Looking at $A > 0$, the term $-gA^3$ means that increasing g, decreases A^*, i.e. the fixed points move towards zero as g increases. Similarly they move away from zero as g decreases. There are therefore 3 fixed points for all values of g. Now we determine their stability properties. As $|A| \to \infty$, we can see that $\dot{A} \approx -kA^5$, meaning $\dot{A} \to \mp\infty$ since $k > 0$. In the neighborhood of $A = 0$ we have $\dot{A} \approx \epsilon A$. Since $\epsilon > 0$, $\dot{A}$ crosses from negative to positive at $A^* = 0$, which is therefore an unstable fixed point. Since the function is continuous this also implies that, at the two other fixed points, $\dot{A}$ goes from positive to negative. $A^* = \pm\sqrt{\alpha_+}$ are therefore two stable fixed points. The stability was derived for arbitrary g, whose value therefore does not affect the nature of the points.

BONUS

(d) Still with $h = 0$, set $\epsilon < 0$ and make a qualitative plot of the bifurcation diagram when g is varied. In in the (k, g) plane, delimit the regions that have a different number of stationary points, finding the analytical equation that identifies these regions.

If $g \geq 0$ the function is monotonically decreasing, the only stationary point is $A^* = 0$, which is stable. On the other hand if g is sufficiently negative and large, $\dot{A}$ is expected to have four more zeros. The boundary between these two cases is given by the condition of the local extrema of $\dot{A}(A)$ to vanish.

Evolution function as g is varied, with $\epsilon < 0$, $h = 0$



In this condition, the function is tangent to the A axis so $\frac{\partial \dot{A}}{\partial A} = 0$ and three stationary points are present, of which two half-stable. In order to find the local extrema where $\frac{\partial \dot{A}}{\partial A} = 0$, we compute $\frac{\partial \dot{A}}{\partial A} = \epsilon - 3gA^2 - 5kA^4$ and introduce $\alpha = A^2 > 0$. We obtain a second order equation.

$$0 = \epsilon - 3g\alpha - 5k\alpha^2$$

that is

$$\alpha^2 + \frac{3g}{5k}\alpha = \frac{\epsilon}{5k}$$

with solution

$$\alpha_\pm = \frac{-3g \pm \sqrt{20\epsilon k + 9g^2}}{10k}$$

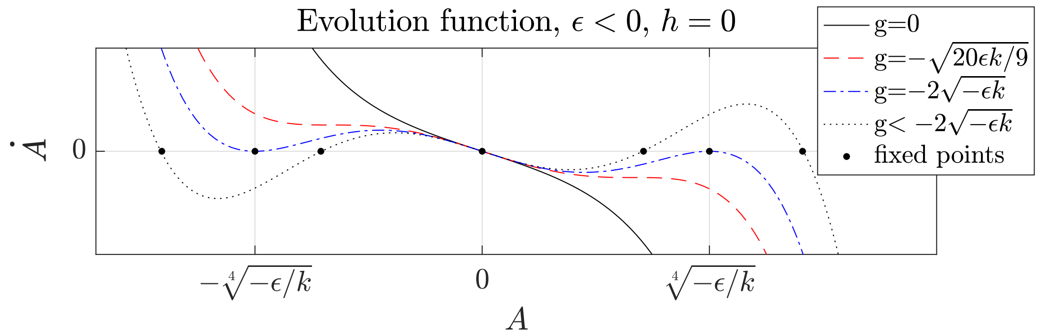
Therefore $A = \pm\sqrt{\alpha_\pm} = \pm\sqrt{\frac{-3g\pm\sqrt{20\epsilon k+9g^2}}{10k}}$ are four points with flat derivatives (local extrema). Notice that they exist only if $20\epsilon k + 9g^2 \geq 0$, i.e. when $g \leq -\sqrt{-\frac{20}{9}\epsilon k}$ (otherwise the function is monotonic). To prove futher that all four are well defined, we must show that $\alpha_\pm > 0$. Indeed $k > 0$ and $g < 0$ (therefore $-3g + \sqrt{20\epsilon k + 9g^2} > 0$). It follows that $\alpha_+ > 0$. Then, for α_-, we notice that with $20\epsilon k < 0$, $20\epsilon k + 9g^2 < 9g^2$, which implies $-\sqrt{20\epsilon k + 9g^2} > 3g$, finally leading to $-3g - \sqrt{20\epsilon k + 9g^2} > 0$. Back to the fixed points, the limiting case when two additional fixed points appear is when the local extrema at $A = \pm\sqrt{\alpha_+}$ touches the A axis :

$$0 = \epsilon\sqrt{\alpha_+} - g\sqrt{\alpha_+}^3 - k\sqrt{\alpha_+}^5$$

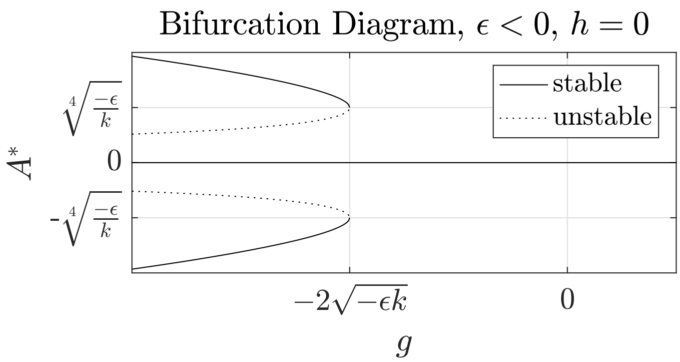
$$0 = \epsilon\sqrt{\frac{-3g + \sqrt{20\epsilon k + 9g^2}}{10k}} - g\sqrt{\frac{-3g + \sqrt{20\epsilon k + 9g^2}}{10k}}^3 - k\sqrt{\frac{-3g + \sqrt{20\epsilon k + 9g^2}}{10k}}^5$$

$$
\begin{aligned}
0 &= \epsilon - g\sqrt{\frac{-3g+\sqrt{20\epsilon k+9g^2}}{10k}}^{2} - k\sqrt{\frac{-3g+\sqrt{20\epsilon k+9g^2}}{10k}}^{4} \\
0 &= \epsilon - g\frac{-3g+\sqrt{20\epsilon k+9g^2}}{10k} - k\left(\frac{-3g+\sqrt{20\epsilon k+9g^2}}{10k}\right)^2 \\
0 &= 10\epsilon k + 3g^2 - g\sqrt{20\epsilon k+9g^2} - \frac{1}{10}\left(9g^2 - 6g\sqrt{20\epsilon k+9g^2} + 20\epsilon k + 9g^2\right) \\
g&\frac{4}{10}\sqrt{20\epsilon k+9g^2} = 8\epsilon k + \frac{12}{10}g^2 \\
g^2&\frac{16}{100}(20\epsilon k+9g^2) = 64\epsilon^2k^2 + \frac{16\cdot 12}{10}\epsilon k g^2 + \frac{12^2}{100}g^4 \\
\frac{2}{10}&g^2 = 4\epsilon k + \frac{12}{10}g^2 \\
g &= -2\sqrt{-\epsilon k}
\end{aligned}
$$

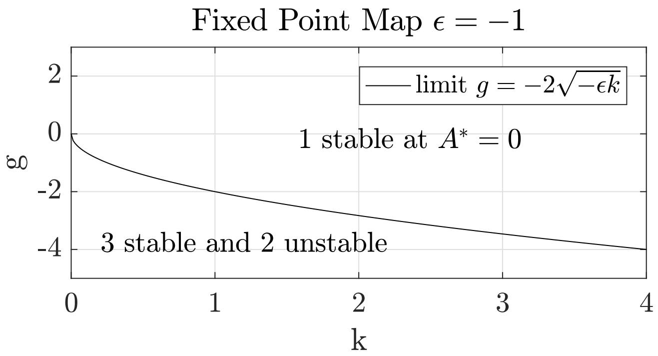
As shown in the figure, at $g = -\sqrt{-20\epsilon k/9}$, the flat derivatives appear. Then at $g = -2\sqrt{-\epsilon k}$ the local extrema touch the A axis. There, three stationary points exist, the stable $A^* = 0$ and the half-stable $A^* = \pm\sqrt{\frac{6\sqrt{-\epsilon k}+\sqrt{20\epsilon k - 36\epsilon k}}{10k}} = \pm\sqrt[4]{\frac{-\epsilon}{k}}$. For larger values of g there are five fixed points.



From the slopes we can tell that the symmetric pair of fixed points closer to the origin is unstable and that the more exterior pair is stable. There are two saddle node bifurcations that occur at $g = -2\sqrt{-\epsilon k}$ and the stable fixed point always at $A^* = 0$.



In the (k, g) plane the curve $g = -2\sqrt{-\epsilon k}$ is the border between the region with a single stable fixed point at the origin and the region with three stable and two unstable fixed points.



Fixed Point Map $\epsilon = -1$
limit $g = -2\sqrt{-\epsilon k}$
1 stable at $A^* = 0$
3 stable and 2 unstable
g
k
2
0
-2
-4
0
1
2
3
4