4 March 2025

Solutions 3 : Bifurcations and Reminder of Linear Algebra

1 Laser Threshold

We model the laser dynamics with the equation $\dot{n} = GnN - kn$, where $n(t) > 0$ is the number of photons in the laser field. In the laser medium the photons decay exponentially due to mirror transmission, scattering, and so on, which explains the $-kn$ term. The photons are created by stimulated emission, when a photon interacts with an excited atom, which de-excites and emits a photon with the same phase, explaining the term $+GnN$. ($N(t) > 0$ is the number of excited atoms, which depends on time, and $G > 0$ is the gain coefficient.) We initially assume that $N = N_0 - \alpha n$, where $N_0 > 0$ is the number of atoms that the pump keeps excited in the absence of photons. The parameter $\alpha > 0$ is a proportionality constant that indicates the decrease of the number of excited atoms because of the presence of photons.

(a) Find the differential equation for n and rescale n so that the laser dynamics depend only on one parameter.

Insert $N(n)$ into the equation gives

$$\dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - G\alpha n^2 \text{ so } \frac{\dot{n}}{n} = (GN_0 - k) - G\alpha n \tag{1}$$

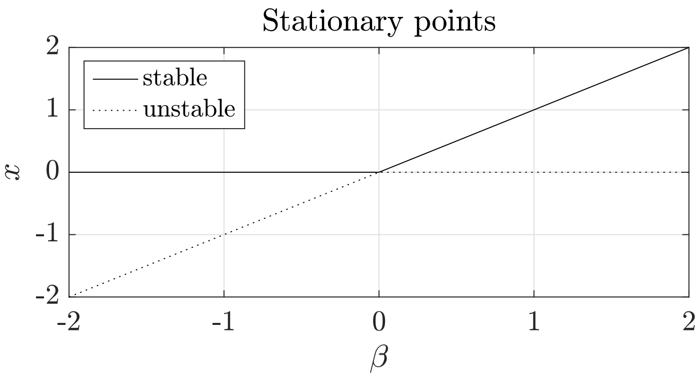
We define $x = G\alpha n$ and $\beta = GN_0 - k$ so that Eq (1) becomes

$$\frac{\dot{x}}{x} = \beta - x \quad \text{that is} \quad \dot{x} = \beta x - x^2 \tag{2}$$

and depends only on the parameter β, which can take any real value.

(b) Find the equilibrium points of n. What are their stability properties ? Identity the bifurcation type as you vary the identified parameter and draw the bifurcation diagram. When are the stationary points physically acceptable ?

The stationary points are solutions to $0 = \beta x^* - x^{*2} \Leftrightarrow x^* = 0$ or $x^* = \beta$. This means $n = 0$ and $n = (GN_0 - k)/(G\alpha)$. The second stationary point only makes physical sense if $\beta > 0$ since the number of photons n is positive. So $\beta > 0 \Rightarrow GN_0 > k$. For the stability, we evaluate $\frac{\partial \dot{x}}{\partial x} = \beta - 2x$. Since $\frac{\partial \dot{x}}{\partial x}|_{x=0} = \beta$, $x^* = 0$ is stable when $\beta < 0$ and becomes unstable when $\beta > 0$. Also $\frac{\partial \dot{x}}{\partial x}|_{x=\beta} = -\beta$, so $x^* = \beta$ is stable when $\beta > 0$ and unstable when $\beta < 0$. As the graph shows, this is a transcritical bifurcation.



(c) What is the threshold condition to emit laser light ?

The pump strength controls N_0. To emit laser light, we need the number of photons n to have a stable positive value. This means $\beta > 0$, i.e. $N_0 > k/G$ is the threshold condition.

To improve the model, the number of excited atoms is assumed to follow $\dot{N} = -GnN - fN + p$. The $-GnN$ is the counterpart to the stimulated emission. $f > 0$ is the decay rate for spontaneous emission. p, positive or negative, is the pump strength. We use the quasi-static approximation $\dot{N} \approx 0$, assuming that N relaxes much faster than n.

(d) Derive the new differential equation for n and renormalise the time and the density so that the model depends only on one parameter.

From
$$\dot{N} = 0 = -GnN - fN + p$$
We have
$$N = \frac{p}{Gn + f}$$
It follows that
$$\dot{n} = Gn\frac{p}{Gn + f} - kn$$
That is
$$\frac{\dot{n}}{k} = n\Big(\frac{Gp/k}{Gn + f} - 1\Big)$$

We introduce the dimensionless time, $kt = \tau$, so that $\frac{1}{k}\dot{n} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}\tau}$. Also we define $x = Gn/f$.

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = n\Big(\frac{Gp/k}{Gn + f} - 1\Big) \Leftrightarrow \frac{\mathrm{d}n}{\mathrm{d}\tau}\frac{G}{f} = \frac{G}{f}n\Big(\frac{Gp/fk}{Gn/f + 1} - 1\Big) \Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}\tau} = x\Big(\frac{\gamma}{x + 1} - 1\Big)$$

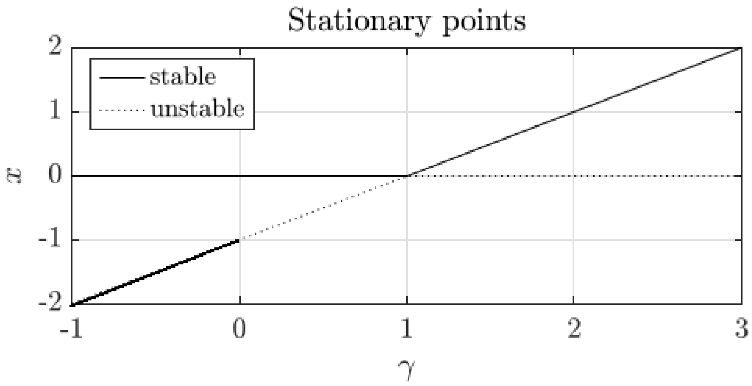
The dynamics depend on only one parameter, $\gamma = \frac{Gp}{fk}$, which is dimensionless and has the same sign as p.

(e) Find the equilibrium points as well as their stability properties. What is the critical pump strength threshold ?

The stationary condition $0 = x^*(\frac{\gamma}{x^*+1} - 1)$ leads to $x^* = 0$ or $0 = \frac{\gamma}{x^*+1} - 1$ that is $x^* = \gamma - 1$, which is physically acceptable when $\gamma > 1$. To determine the stability, we differentiate

$$\frac{\partial}{\partial x}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right) = \gamma\frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} - 1 = \frac{\gamma}{(x+1)^2} - 1$$

At $x^* = 0$, we have $\frac{\partial}{\partial x}(\frac{\mathrm{d}x}{\mathrm{d}\tau}) = \gamma - 1$. At $x^* = \gamma - 1$, it is $\frac{\partial}{\partial x}(\frac{\mathrm{d}x}{\mathrm{d}\tau}) = \frac{1}{\gamma} - 1$. The bifurcation diagram is very similar to the one evaluated in question (b).



The critical pump strength is such that $1 = \gamma = \frac{Gp}{fk} \Leftrightarrow p = \frac{fk}{G}$.

2 Laser threshold with an advanced model

For a laser model considerably improved with respect to the one considered in the first exercise, we introduce the electric field E, the mean polarisation P and the population inversion D. They follow the Maxwell-Bolch equations :

$$\begin{cases} \dot{E} = \kappa(P - E) \\ \dot{P} = \gamma_1(ED - P) \\ \dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP) \end{cases}$$

The constant $\kappa > 0$ is the decay rate in the laser cavity due to beam transmission. $\gamma_1, \gamma_2 > 0$ are the decay rates of the atomic polarization and population inversion. λ, positive or negative, is the pumping strength. If $\gamma_1, \gamma_2 \gg \kappa$ then the adiabatic elimination can be used, where $\dot{P} = 0$ and $\dot{D} = 0$. Using the adiabatic approximation, simplify the evolution equation of E. Then, plot the bifurcation diagram of E with respect to the parameter λ.

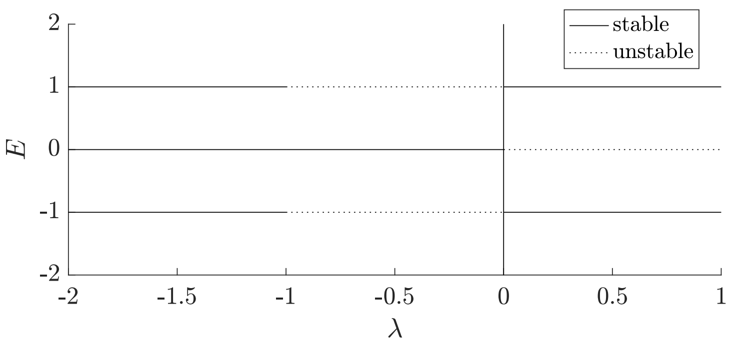
The adiabatic approximation gives $\dot{P} = 0 = \gamma_1(ED - P)$ that is $P = ED$. Similarly $\dot{D} = 0 = \gamma_2(\lambda + 1 - D - \lambda EP)$ implies $0 = \lambda + 1 - D - \lambda E^2 D$ that is $D = \frac{\lambda+1}{1+\lambda E^2}$. So the important result is

$$\dot{E} = \kappa(E\frac{\lambda + 1}{1 + \lambda E^2} - E) = \kappa E\frac{\lambda + 1 - 1 - \lambda E^2}{1 + \lambda E^2} = \kappa E\lambda\frac{1 - E^2}{1 + \lambda E^2}$$

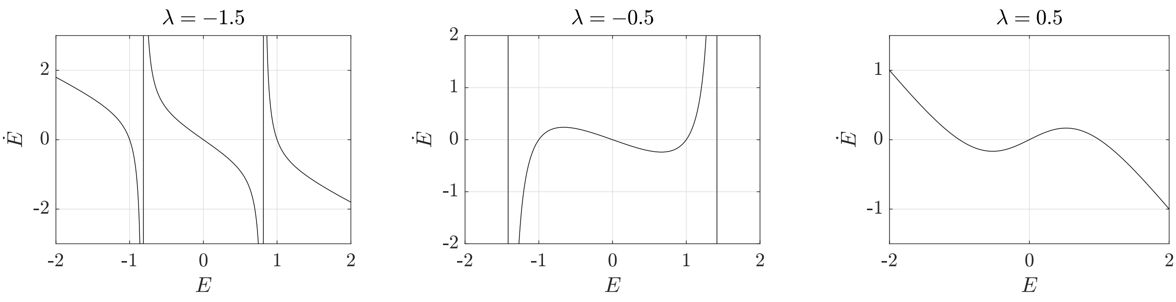
Using the dimensionless time $\tau = \kappa t$ gives $\frac{\mathrm{d}E}{\mathrm{d}\tau} = \lambda E\frac{1-E^2}{1+\lambda E^2}$. The stationary points are $E^* = 0$ and $0 = 1 - E^{*2}$ that is $E^* = \pm 1$. Their stability depend on the sign of

$$\begin{aligned} \frac{\partial}{\partial E}\left[\frac{\mathrm{d}E}{\mathrm{d}\tau}\right] &= \lambda\frac{(1 - 3E^2)(1 + \lambda E^2) - (E - E^3)2\lambda E}{(1 + \lambda E^2)^2} \\ &= \lambda\frac{1 - 3E^2 + \lambda E^2 - 3\lambda E^4 - 2\lambda E^2 + 2\lambda E^4}{(1 + \lambda E^2)^2} \\ &= \lambda\frac{1 - 3E^2 - \lambda E^2 - \lambda E^4}{(1 + \lambda E^2)^2} \end{aligned}$$

At $E^* = 0$ we have $\frac{\partial}{\partial E}\left[\frac{\mathrm{d}E}{\mathrm{d}\tau}\right] = \lambda\frac{1-0-0-0}{(1+0)^2} = \lambda$, stable when $\lambda \leq 0$ and ustable when $\lambda > 0$. At $E^* = \pm 1$ we have $\frac{\partial}{\partial E}\left[\frac{\mathrm{d}E}{\mathrm{d}\tau}\right] = \lambda\frac{1-3-\lambda-\lambda}{(1+\lambda)^2} = \lambda\frac{-2-2\lambda}{(1+\lambda)^2} = \frac{-2\lambda}{1+\lambda}$. This is stable except when $\lambda \in]-1, 0[$. Note that at $\lambda = 0$, the differential equation becomes $\dot{E} = 0$ so all values of E are stationary points. The bifurcation diagram is therefore



There are three regions where the function $\dot{E}(E)$ has qualitatively different shapes and consequently different stability properties of the fixed point. These plots show these different characteristic shapes.


$\lambda = -1.5$
$\lambda = -0.5$
$\lambda = 0.5$
$\dot{E}$
E
2
0
-2
-2
-1
0
1
2
1
-1

3 Exponential of a Matrix and Systems of Linear Differential Equations

As a warm up for the analysis of multi-dimensional non-linear systems, we recall how multi-dimensional linear systems can be solved and how the exponential of a matrix can be evaluated.

(a) A square matrix N is nilpotent if there exists $m_* \in \mathbb{N}$ such that $N^m = 0,\ \forall m > m_*$. Compute the powers of the upper shift matrix of size 4, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Use a computer if necessary. Notice the pattern ? Is this a nilpotent matrix ?

We can see that it is a nilpotent matrix, as the diagonal of ones moves up and disappears.

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the exponential of the upper shift matrix N.
Reminder : For matrices, $\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!} = I + M + \frac{1}{2}M^2 + \dots$

$$\exp(N) = I + N + \frac{1}{2}N^2 + \frac{1}{6}N^3 + 0 = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) What is the exponential of a diagonal matrix ?

$$\exp(D) = \sum_{n=0}^{\infty} \frac{D^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}$$

(d) Compute the exponential of a Jordan block J_λ, that is

$$\exp(J_\lambda) = \exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

Hint : If two matrices, A and B, commute (i.e. $[A, B] = AB - BA = 0$), then $\exp(A + B) = \exp(A)\exp(B)$.

Since $J_\lambda = \lambda I + N_3$ and $[\lambda I, N_3] = 0 \ \Rightarrow$

$$\exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \exp\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \exp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} e^\lambda & 0 & 0 \\ 0 & e^\lambda & 0 \\ 0 & 0 & e^\lambda \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^\lambda & e^\lambda & e^\lambda/2 \\ 0 & e^\lambda & e^\lambda \\ 0 & 0 & e^\lambda \end{bmatrix}$$

(e) Show that the matrix

$$M = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

can be written as a Jordan block under a basis transformation and evaluate the proper transition matrix.

Reminder : Suppose a 3 by 3 matrix M can be written as J_λ by changing basis, i.e. there exists a transition matrix P such that $M = PJ_\lambda P^{-1}$. P is the 3×3 matrix whose columns are the three new basis vectors side by side. To find the basis vectors notice that if M, represented in the $\{v_1, v_2, v_3\}$ basis, is written

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

then, by definition,

$$\begin{cases} Mv_1 = \lambda v_1 \\ Mv_2 = \lambda v_2 + v_1 \\ Mv_3 = \lambda v_3 + v_2 \end{cases}$$

Indeed, the basis vectors can be found by solving

$$\begin{cases} (M - \lambda I)v_1 = 0 \\ (M - \lambda I)v_2 = v_1 \\ (M - \lambda I)v_3 = v_2 \end{cases}$$

To find the eigenvector we solve $M - 3I = 0$.

$$(M - 3I)v_1 = 0 \Leftrightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 2y - z = 0 \\ 0 = 0 \\ -y = 0 \end{cases} \Leftrightarrow \begin{cases} z = 0 \\ 0 = 0 \\ y = 0 \end{cases}$$

We can choose $v_1 = e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, as one could have seen directly when looking at the matrix M. To find the second basis vector, solve $(M - 3I)v_2 = v_1$.

$$(M - 3I)v_2 = v_1 \Leftrightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 2y - z = 1 \\ 0 = 0 \\ -y = 0 \end{cases} \Leftrightarrow \begin{cases} z = -1 \\ 0 = 0 \\ y = 0 \end{cases}$$

Indeed any vector of the form $\begin{bmatrix} x & 0 & -1 \end{bmatrix}^T$ does the job. We can choose for example $v_2 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^T$. Finally, we find v_3.

$$(M - 3I)v_3 = v_2 \Leftrightarrow \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \Leftrightarrow \begin{cases} 2y - z = 0 \\ 0 = 0 \\ -y = -1 \end{cases} \Leftrightarrow \begin{cases} z = 2 \\ 0 = 0 \\ y = 1 \end{cases}$$

Again any vector $\begin{bmatrix} x & 1 & 2 \end{bmatrix}^T$ does the job. Choose the simplest one $v_3 = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}^T$. The transition matrix P is just $[v_1, v_2, v_3]$ side by side, so :

$$M = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = PJ_3P^{-1}$$

(f) Using the properties of the matrix exponential, explicitly compute the solution of the system $\dot{\vec{x}} = M\vec{x}$, with $\vec{x}(t_0) = \vec{x}_0 = [x_{01}, x_{02}, x_{03}]^T$.
<u>*Reminder*</u> : $\exp(A\ B\ A^{-1}) = A\ \exp(B)\ A^{-1}$

The solution is $\vec{x}(t) = \exp[M(t-t_0)]\vec{x}_0$. Thus we need $\exp[M\tau]$ (with $\tau = t - t_0$). To do so, first compute $\exp[J_3\tau]$

$$\exp\left(\begin{bmatrix}3 & 1 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{bmatrix}\tau\right) = \exp\left(\begin{bmatrix}3\tau & 0 & 0\\ 0 & 3\tau & 0\\ 0 & 0 & 3\tau\end{bmatrix} + \begin{bmatrix}0 & \tau & 0\\ 0 & 0 & \tau\\ 0 & 0 & 0\end{bmatrix}\right)$$

The two matrices commute, since the first one is a scalar matrix.

$$\begin{aligned}
&\exp\left(\begin{bmatrix}3\tau & 0 & 0\\ 0 & 3\tau & 0\\ 0 & 0 & 3\tau\end{bmatrix}\right)\cdot\exp\left(\begin{bmatrix}0 & \tau & 0\\ 0 & 0 & \tau\\ 0 & 0 & 0\end{bmatrix}\right)\\
&= \begin{bmatrix}e^{3\tau} & 0 & 0\\ 0 & e^{3\tau} & 0\\ 0 & 0 & e^{3\tau}\end{bmatrix}\cdot\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0 & \tau & 0\\ 0 & 0 & \tau\\ 0 & 0 & 0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 0 & \tau^2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right)\\
&= e^{3\tau}\begin{bmatrix}1 & \tau & \tau^2/2\\ 0 & 1 & \tau\\ 0 & 0 & 1\end{bmatrix}
\end{aligned}$$

Pluging this result in gives

$$\begin{aligned}
\exp(M\tau) = P\exp(J_3\tau)P^{-1} &= \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 2\end{bmatrix} e^{3\tau}\begin{bmatrix}1 & \tau & \tau^2/2\\ 0 & 1 & \tau\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 2\end{bmatrix}^{-1}\\
&= e^{3\tau}\begin{bmatrix}1 & 2\tau + \tau^2/2 & -\tau\\ 0 & 1 & 0\\ 0 & -\tau & 1\end{bmatrix}
\end{aligned}$$

Therefore the solution is

$$\vec{x}(t) = \begin{bmatrix}[x_{01} + (2(t-t_0) + (t-t_0)^2/2)x_{02} - (t-t_0)x_{03}]e^{3(t-t_0)}\\ x_{02}e^{3(t-t_0)}\\ [-(t-t_0)x_{02} + x_{03}]e^{3(t-t_0)}\end{bmatrix}$$

<u>**A general reminder on the exponentials of matrices**</u> :
Any matrix M can be written in the Jordan normal form $M = PJP^{-1}$, with J composed of Jordan blocks like equation (3) and P a transition matrix. The number of Jordan blocks associated with the eigenvalue λ_n is the number of eigenvectors that M has for this eigenvalue. For example, consider

$$J = \begin{bmatrix}\lambda_1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \lambda_3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & 1 & 0 & & & & \\ 0 & \lambda_1 & 1 & & & & \\ 0 & 0 & \lambda_1 & & (0) & & \\ & & & \lambda_1 & & & \\ & & & & \lambda_2 & & \\ & & (0) & & & \lambda_3 & 1\\ & & & & & 0 & \lambda_3\end{bmatrix} \quad (3)$$

For λ_1, there are two eigenvectors (There is a Jordan block of size 3, and a second block of size 1.) For λ_2, there is one eigenvector. For λ_3, there is also only one eigenvector (we have a Jordan block of size 2).

To find the basis in which M is written as J one has to :

— Find all the eigenvalues and as many linearly independent eigenvectors as possible.

— If there is only one eigenvector v_λ, but the algebraic multiplicity of the eigenvalue is $g_\lambda > 1$, find other generalized eigenvectors by solving $(M - \lambda I)v_\lambda^{(2)} = v_\lambda$, then $(M - \lambda I)v_\lambda^{(3)} = v_\lambda^{(2)}$ and so on until you have g_λ generalized eigenvectors.

— If there are multiple eigenvectors for the same eigenvalue $\{v_{\lambda,1}, v_{\lambda,2}, \ldots, v_{\lambda,n}\}$, but the algebraic multiplicity of the eigenvalue is $g_\lambda > n$, then one has to find the eigenvectors $v_{\lambda,*} = \sum_i a_i v_{\lambda,i}$ that are in the image of $M - \lambda I$. Only for these eigenvectors there exists rank 2 generalized eigenvectors satisfying $(M - \lambda I)v_{\lambda,*}^{(2)} = v_{\lambda,*}$. Continue similarly to find the rank 3 generalized eigenvectors until you have g_λ generalized eigenvectors.

As an example, let us find the Jordan normal form of matrix M, whose only eigenvalue is 2, as well as a transition matrix P.

$$M_2 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

First, we find an eigenvector basis.

$$(M_2 - 2I)v = 0 \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} 0 = 0 \\ -x + 2y - 2z = 0 \\ -x + 2y - 2z = 0 \end{cases} \Leftrightarrow x = 2(y - z)$$

A possible basis is $\{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T\}$. But since the image of $M_2 - 2I$ is $\mathrm{Span}(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T)$, a proper basis to be used is $v_1 = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$. Note that indeed $v_2 - v_1 = \begin{bmatrix} -2 & 0 & 1 \end{bmatrix}^T$. Now we find the generalized eigenvector that gives the Jordan normal form.

$$(M_2 - 2I)v_3 = v_2 \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{cases} 0 = 0 \\ -x + 2y - 2z = 1 \\ -x + 2y - 2z = 1 \end{cases} \Leftrightarrow x = 2(y - z) - 1$$

We can take $\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}^T$, which gives

$$M_2 = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -2 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = PJP^{-1}$$

Now, let us compute $\exp(M_3)$, with

$$M_3 = \begin{bmatrix} 3 & 2 & -1 & & & \\ 0 & 3 & 0 & & (0) & \\ 0 & -1 & 3 & & & \\ & & & 2 & 0 & 0 \\ & (0) & & -1 & 4 & -2 \\ & & & -1 & 2 & 0 \end{bmatrix}$$

We use Mathematica to compute the inverse matrix P^{-1} and the matrix multiplications. Blocks can be treated separately. Using the previous exercises

$$M_3 = PJP^{-1} = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & (0) & \\ 0 & -1 & 2 & & & \\ & & & 2 & 0 & -1 \\ & (0) & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 & & & \\ 0 & 3 & 1 & & (0) & \\ 0 & 0 & 3 & & & \\ & & & 2 & 0 & 0 \\ & (0) & & 0 & 2 & 1 \\ & & & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & (0) & \\ 0 & -1 & 2 & & & \\ & & & 2 & 0 & -1 \\ & (0) & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix}^{-1}$$

So the exponential is

$$\exp(M_3) = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & (0) & \\ 0 & -1 & 2 & & & \\ & & & 2 & 0 & -1 \\ & (0) & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^3 & e^3 & e^3/2 & & & \\ 0 & e^3 & e^3 & & (0) & \\ 0 & 0 & e^3 & & & \\ & & & e^2 & 0 & 0 \\ & (0) & & 0 & e^2 & e^2 \\ & & & 0 & 0 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & (0) & \\ 0 & -1 & 2 & & & \\ & & & 2 & 0 & -1 \\ & (0) & & 1 & 1 & 0 \\ & & & 0 & 1 & 0 \end{bmatrix}^{-1}$$

That is

$$\exp(M_3) = \begin{bmatrix} e^3 & \frac{5e^3}{2} & -e^3 & & & \\ 0 & e^3 & 0 & & (0) & \\ 0 & -e^3 & e^3 & & & \\ & & & e^2 & 0 & 0 \\ & (0) & & -e^2 & 3e^2 & -2e^2 \\ & & & -e^2 & 2e^2 & -e^2 \end{bmatrix}$$

On Mathematica this can be done very quickly using `MatrixExp`.