13May 2024

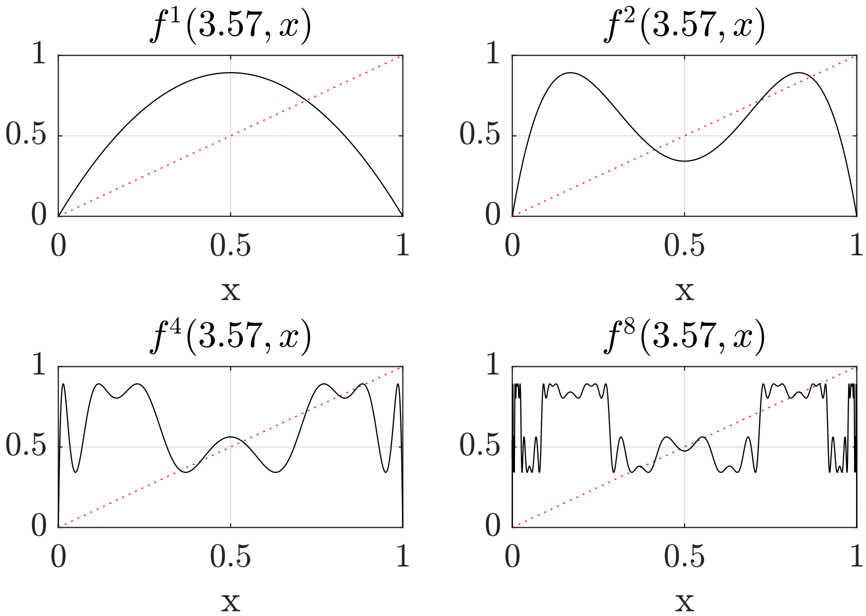
Solutions 11 : Renormalisation

1 Renormalised Plots

The goal of this exercise is to visualise that the functions $f^{2^r}(A_r, x)$ can be renormalised to look very similar. Consider the logistic equation $f(a, x) = ax(1 - x)$.

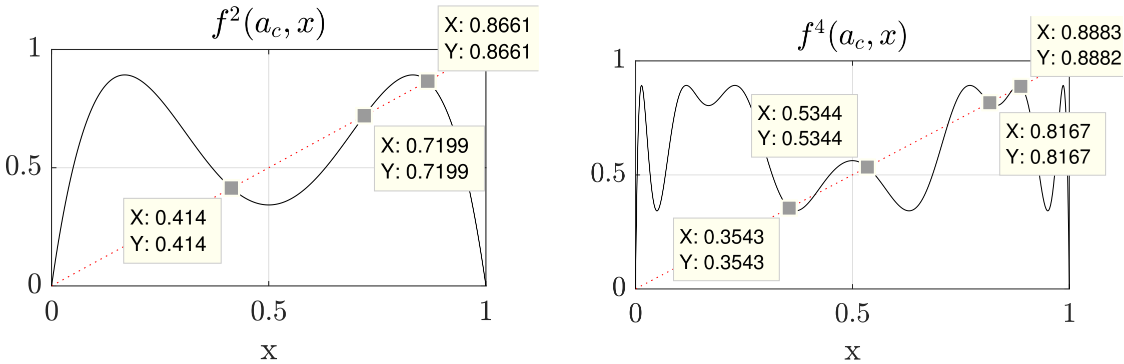
(a) First consider $a = a_c \approx 3.5699$ (at $a = a_c$ the system becomes chaotic). Plot four graphs $y = f^{2^r}(a_c, x)$, on interval $x \in [0, 1]$, with $r \in \{0, 1, 2, 3\}$.

The code is the `animation_f2r.m` file. Since this is question (a), call the function with argument as such `animation_f2r('a')`. The code is reused for later questions.



(b) Add the diagonal $y = x$ on the plots. The intersections where $x = f^n(a_c, x)$ correspond to fixed points, which are all unstable since $a = a_c$. A fixed point of f^n corresponds to a point of an m-cycle, where m is a divisor of n. On the graph of f^2, identify the fixed points and the period of the cycle associated. Try to identify the 4-cycle using the graph of f^4.

We can see on the f^1 graph that $x = 0$ and $x \approx 0.72$ are fixed points, or part of a 1-cycles. On the f^2 graph there are four fixed points. The first two are again $x = 0$ and $x \approx 0.72$, which form a 1-cycles. The other two form a 2-cycle at $x \approx 0.41$ and $x \approx 0.87$.



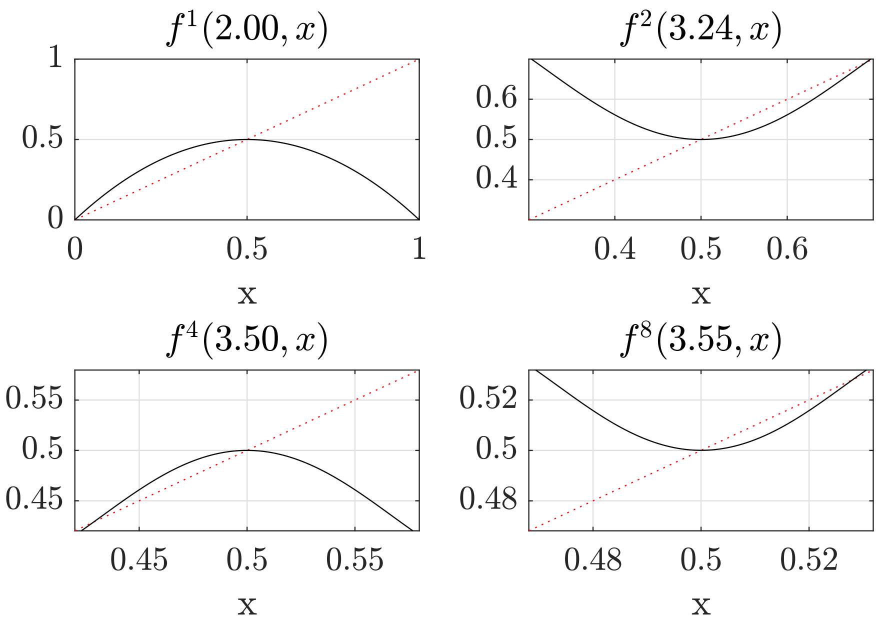
The 4-cycle of f^4 appears from the flip bifurcations which create fixed points on either side of the points of the 2-cycle. The figure shows the 4-cycle, which goes through $\{0.35, 0.82, 0.53, 0.88\}$.

(c) Now animate the plots of f^{2^r}, $r \in \{0, 1, 2, 3\}$ by varying a. For all four graphs consider the range $0 \leq a \leq a_c$. Identify the flip bifurcations and period doubling.

For the animation, call the function as such `animation_f2r('c')`. The flip bifurcations can be identified as the slope of the fixed point increases and becomes greater than one. It means that the associated cycle becomes unstable, and that two new fixed points appear around it.

(d) The graphs f^{2^r} all look the same near their associated A_r, close to the maximum at $x = 0.5$. To visualise it, animate again the graphs, this time for f^{2^0} use the range $1 \leq a \leq A_0$, and in general $A_{r-1} \leq a \leq A_r$ for f^{2^r}. Also magnify the plots as r increases by using a plot range centered at $(x, y) = (0.5, 0.5)$ that is a square of side $1/\alpha^r$.
Help : Use $A_0 = 2$, $A_1 = 3.2361$, $A_2 = 3.4986$, $A_3 = 3.5546$, $A_4 = 3.5667$.

For the animation, call the function as such `animation_f2r('d')`. The range used in both x and y to zoom in is $[\frac{1}{2} - \frac{1}{2}\frac{1}{|\alpha|^r}, \frac{1}{2} + \frac{1}{2}\frac{1}{|\alpha|^r}]$. We can see that the graphs for different r values look very similar when we zoom in. This is why the renormalised functions converge. Also notice that they alternate upside down at each iteration of r. Indeed the renormalisation multiplies by $\alpha < 0$.



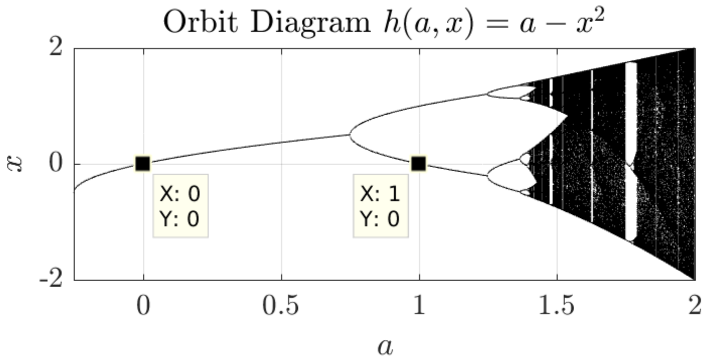
2 Universal Constant

Consider the function $f(a, x) = a - (x - 2)^2$ which has a quadratic maximum.

(a) Find $x = x_m$, the location of the maximum of f. Then, find $h(a, x)$, the translation of f with maximum at $x = 0$.

Since $\frac{\partial f}{\partial x}(a, x) = -2(x - 2)$, the maximum is reached at $x_m = 2$. The function $h(a, x)$ is therefore $h(a, x) = f(a, x_m + x) = a - (x_m + x - 2)^2 = a - x^2$.

(b) Numerically plot the orbit diagram of h for $-1/4 \leq a \leq 2$, using $x_0 = 0$. Graphically determine A_r, with $r = \{0, 1\}$, the values of a at which the 2^r-cycle has a point that coincides with the maximum of h.



The orbit diagram shows again the period-doubling route to chaos we are used to for functions with a quadratic maximum. On the figure we identify $A_0 = 0$ and $A_1 = 1$.

(c) Identify the explicit expressions of the functions $g_{0r}(x) = \alpha^r h^{2^r}(A_r, \frac{x}{\alpha^r})$, for $r = \{0, 1\}$.

First one is

$$g_{00}(x) = \alpha^0 h^1\Big(A_0, \frac{x}{\alpha^0}\Big) = h(0, x) = -x^2$$

The second one is

$$\begin{aligned} g_{01}(x) &= \alpha h^2\Big(A_1, \frac{x}{\alpha}\Big) = \alpha h\Big(1, \alpha\frac{h(1, \frac{x}{\alpha})}{\alpha}\Big) \\ &= \alpha h\Big(1, 1 - \frac{x^2}{\alpha^2}\Big) = \alpha\Big[1 - \Big(1 - \frac{x^2}{\alpha^2}\Big)^2\Big] \\ &= \alpha\Big[1 - \Big(1 - \frac{2x^2}{\alpha^2} + \frac{x^4}{\alpha^4}\Big)^2\Big] \\ g_{01}(x) &= \frac{2x^2}{\alpha} - \frac{x^4}{\alpha^3} \end{aligned}$$

(d) From what we have learned in class, the functions g_{0r}, which are renormalised using the universal constant α, should resemble each other near $x = 0$, and should converge to a function g_0 as $r \to \infty$. Keeping the lowest-order terms, what should be the value of α so that $g_{00} \approx g_{01}$ near $x = 0$? How does this compare with the Feigenbaum constant?

Near $x = 0$, we can only keep the lowest order terms, so $g_{00}(x) \approx -x^2$ and $g_{01}(x) \approx \frac{2x^2}{\alpha}$. The first order terms are the same for $\alpha = -2$. We know that $\alpha \approx -2.5$, this is indeed the right order of magnitude.

(e) Using Mathematica, show that A_2 is the root of a 8^{th} order polynomial, and numerically find it.

A_2 is a the value of a for which 0 is a fixed point of h^4. This means $h^4(A_2, 0) = 0$, therefore A_2 is a root of $h^4(A_2, 0)$. On Mathematica use the commands

```
h[a_,x_]:=a-x∧2
h4[a_,x_]:=h[a,h[a,h[a,h[a,x]]]]
Expand[h4[a, 0]]
```

The computer returns `a -a∧2 +2a∧3 -5a∧4 +6a∧5 -6a∧6 +4a∧7 -a∧8`. To get the roots use the command

```
NSolve[ h4[a,0]==0, a]
```

Among the solutions there is $a = 0$ and $a = 1$, which correspond to A_0 and A_1, since $h(A_0, 0) = 0$ implies $h^4(A_0, 0) = 0$. The appropriate solution is $A_2 \approx 1.3107$.

(f) Using Mathematica, express the lowest order term of $g_{02}(x)$, which is proportional to x^2, as a function of α and A_2. Then, use the numerical value of A_2 to compute the value of α that would make the lowest-order terms of g_{01} and g_{02} equal.

On Mathematica use command

```
Expand[ α∧2 h4[A2, x/α∧2] ]
```

which returns a big polynomial. At first it may look like the lowest order is $\mathcal{O}(1)$, since it is $(A_2 - A_2^2 + 2A_2^3 - 5A_2^4 + 6A_2^5 - 6A_2^6 + 4A_2^7 - A_2^8)\alpha^2$, but actually the value of A_2 cancels this term by definition. This makes $(8A_2^3 - 16A_2^4 + 24A_2^5 - 24A_2^6 + 8A_2^7)x^2/\alpha^2$ be the lowest-order term. If we suppose that g_{01} and g_{02} have the same coefficient at the lowest order (the g_{0r} converge to the same g_0 as $r \to \infty$), we have

$$\frac{8A_2^3 - 16A_2^4 + 24A_2^5 - 24A_2^6 + 8A_2^7}{\alpha^2} = \frac{2}{\alpha}$$

$$\alpha = \frac{8A_2^3 - 16A_2^4 + 24A_2^5 - 24A_2^6 + 8A_2^7}{2} \approx -2.44433$$

This approximation for α is closer than the previous one.

3 Quartic Chaos

Up to now we have seen the path to chaos around the quadratic maximum of a map f. The renormalisation was carried out by using the universal constants α and δ. These universal constants are different if we study a map presenting a quartic maximum. Similarly to the quadratic maximum, for a function f with a quartic maximum at $x = 0$ we can renormalise $g_{qr}(x) = \alpha^r f^{2^r}(A_{r+q}, \frac{x}{\alpha^r})$. Then, $g_q(x) = \lim_{r\to\infty} g_{qr}(x)$ and the universal function $g(x) = \lim_{q\to\infty} g_q(x)$ has the property $g(x) = \alpha g(g(\frac{x}{\alpha}))$.

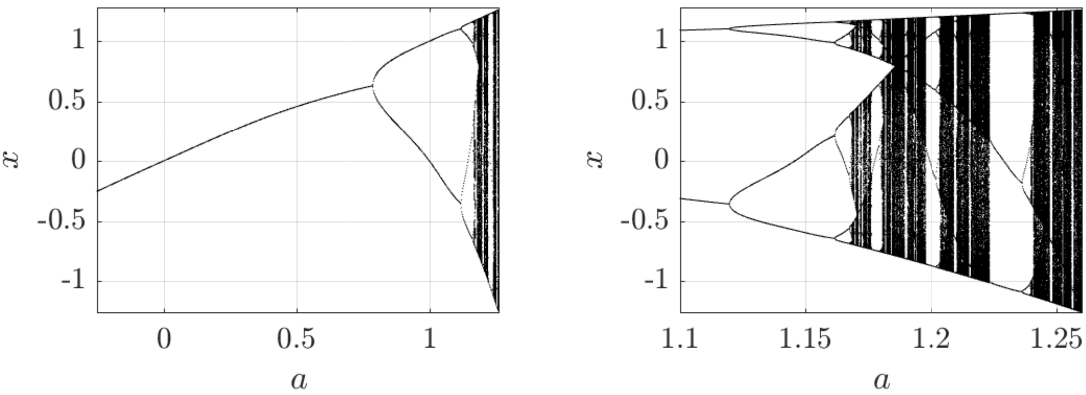
(a) We expect $g(x)$ to behave like $g_{00}(x) = f(A_0, x)$ near $x = 0$. Suppose therefore that $g(x) = 1 + bx^4$ and, neglecting the higher-order terms, use the equation $g(x) = \alpha g(g(\frac{x}{\alpha}))$ to determine an approximate value of b and α.

$$\begin{aligned} 1 + bx^4 &= \alpha\Big(1 + b\Big(1 + b\frac{x^4}{\alpha^4}\Big)^4\Big) \\ &= \alpha\Big(1 + b\Big(1 + 4b\frac{x^4}{\alpha^4}\Big)\Big) + \mathcal{O}(x^8) \\ 1 + bx^4 &= \alpha(1 + b) + 4b^2\frac{x^4}{\alpha^3} + \mathcal{O}(x^8) \end{aligned}$$

Equating the coefficient of the lowest orders gives $1 = \alpha(1+b)$ and $b = 4b^2/\alpha^3$. Use the first equation to get $\alpha = \frac{1}{1+b}$ then plug it into the second equation to get $b = 4b^2(1 + b)^3$. Mathematica can solve this equation, giving multiple solutions. Since $g_{00}(x) = f(A_0, x)$ has a maximum at $x = 0$, then b is real and negative. The only solution left is $b \approx -1.54493$ which implies $\alpha \approx -1.83509$. The more precise value found by Briggs (1991) is $\alpha \approx -1.69030$.

(b) Numerically plot the orbit diagram of $f(a, x) = a - x^4$ for $-0.25 \le a \le 1.26$ using $x_0 = 0$.

The plots can be generated with the same code given in previous problem sets. We observe that the quartic orbit diagram is qualitatively similar to the quadratic one. Quantitatively though it can be seen that the distances are not the same, for example the relative size of the gaps (like where the 3-cycle appears) is bigger.



(c) Now, compute an approximate value of δ from A_0, A_1 and A_2.
Help : Use the command `NSolve` in Mathematica and do not go any further than A_2 as computations get very complicated.

On Mathematica use commands

```
f[a_,x_] := a - x∧4
f2[a_,x_] := f[a, f[a, x]]
f4[a_,x_] := f2[a, f2[a, x]]
Solve[f[a, 0] == 0, a]
Solve[f2[a, 0] == 0, a, Reals]
NSolve[f4[a, 0] == 0, a, Reals]
```

The specification `Reals` means that it only looks for real solutions. We obtain $A_0 = 0$, $A_1 = 1$ (which we could both see on the orbit diagram) and $A_2 \approx 1.1457$. We do not go further, since $f^4(a, 0)$ is a polynomial function of a of order 64. The approximation we can make is therefore $\delta \approx \frac{A_1 - A_0}{A_2 - A_1} \approx 6.86275$. The more precise value found by Briggs (1991) is $\delta \approx 7.28469$.