18 February 2025

Solutions 1 : The Lorenz Model

1 Numerical Approach to the Lorenz Model

We have seen in class that the Lorenz equations are :

$$\begin{cases} \dot{X} = \sigma(Y - X) \\ \dot{Y} = -XZ + rX - Y \\ \dot{Z} = XY - bZ \end{cases}$$

(a) Using Matlab, create a program that integrates the Lorenz equations for given initial conditions. Compare the explicit Euler and Runge-Kutta 4$^{\text{th}}$ order algorithms as well as Matlab's `ode45` function. Use initial condition $(X_0, Y_0, Z_0) = (0, 1, 0)$, parameters $\sigma = 10$, $r = 10$, $b = 8/3$, intitial time $t_0 = 0$, final time $t_f = 2$ and time step $\Delta t = 0.01$.

Reminders :

For the differential equation $\dot{\vec{x}} = \vec{f}(t, \vec{x})$, the explicit Euler method approximates

$$\vec{x}(t + \Delta t) \approx \vec{x}(t) + \Delta t \cdot \vec{f}(t, \vec{x}(t))$$

while Runge-Kutta 4$^{\text{th}}$ order approximates

$$\vec{x}(t + \Delta t) \approx \vec{x}(t) + \frac{1}{6}\left[\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right]$$

with

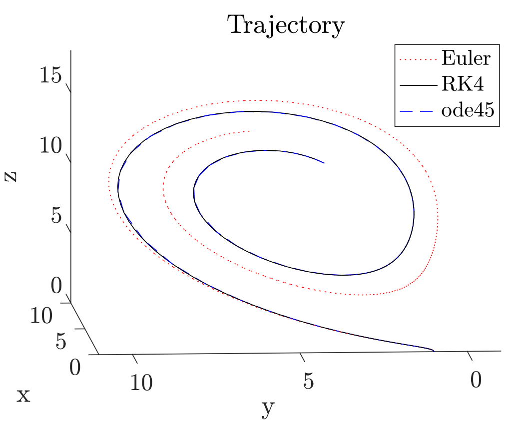
$$\begin{aligned} \vec{k}_1 &= \Delta t \cdot \vec{f}(t, \vec{x}(t)) \\ \vec{k}_2 &= \Delta t \cdot \vec{f}\Big(t + \frac{\Delta t}{2}\ ,\ \vec{x}(t) + \frac{1}{2}\vec{k}_1\Big) \\ \vec{k}_3 &= \Delta t \cdot \vec{f}\Big(t + \frac{\Delta t}{2}\ ,\ \vec{x}(t) + \frac{1}{2}\vec{k}_2\Big) \\ \vec{k}_4 &= \Delta t \cdot \vec{f}\Big(t + \Delta t\ ,\ \vec{x}(t) + \vec{k}_3\Big) \end{aligned}$$

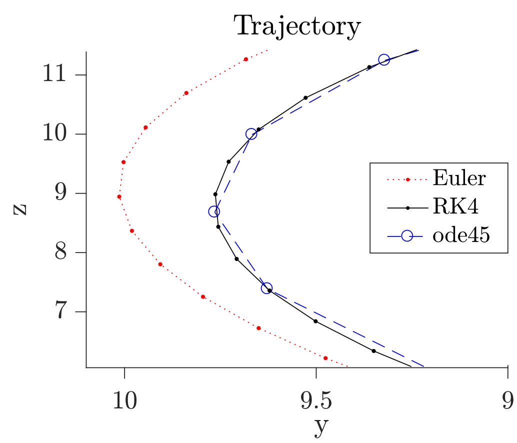
We also note that for Matlab's `ode45` the command line is `ode45(@f,[t0 tf],X0)`, where :

— `t0` and `tf` are the initial and final times

— `X0` the *column* vector containing the initial conditions $\vec{x}(t = t_0)$

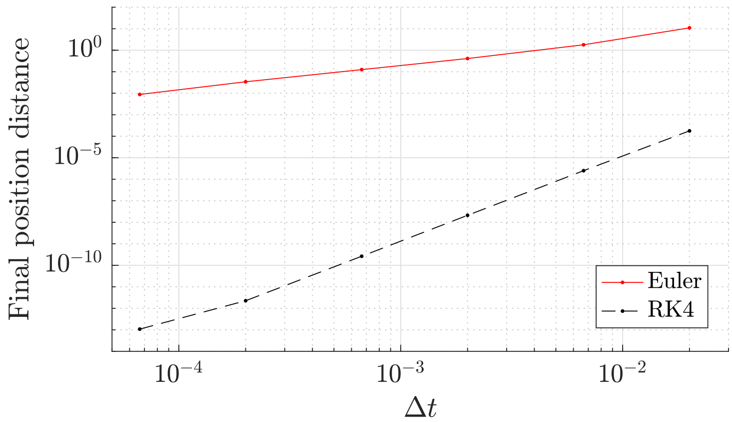
— `f` is a function whose arguments are the time t and the *column* vector $\vec{x}$. It returns the *column* vector $\vec{f}(t, \vec{x})$





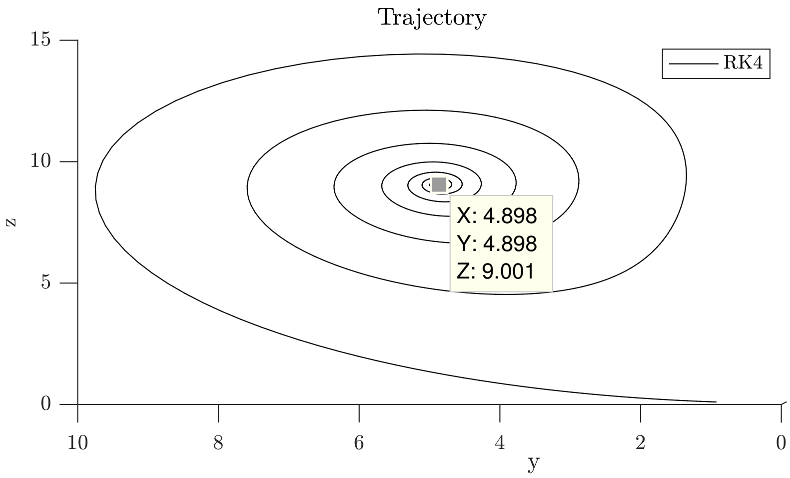
See the matlab code for solutions. The Euler solution visibly separates even before the first oscillation. The Runge-Kutta 4 and `ode45` follow each other pretty well. Zooming in on the trajectories, it can be seen that `ode45` manages to get the same result with a bigger time step as we observe sharp angles.

(b) Now refine the time step to show that your Euler and Runge-Kutta methods converge according to the order of the method. To do so, run simulations with different Δt. For the two schemes, consider the final position $\vec{x}(t_f)$ of the simulation with the smallest Δt to be your exact solution. Plot the error on the final state $||\vec{x}_{\text{exact}}(t_f) - \vec{x}(t_f)||_2$ for the various time steps.



On this logarithmic graph, the slope is the order of convergence of the schemes. No surprise, Euler converges in 1^{rst} order and Runge-Kutta in 4^{th} order.

(c) It is in general very difficult to find an analytical solution for the nonlinear system $\dot{\vec{x}} = \vec{f}(\vec{x})$. However, if there is an $\vec{x}^*$ such that $\vec{f}(\vec{x}^*) = 0$ then $\vec{x}(t) = \vec{x}^*$ is an analytical stationary solution. The points $\vec{x}^*$ that give $\vec{f}(\vec{x}^*) = 0$ are called equilibrium or fixed points. Analytically find the equilibrium points of the Lorenz system. With the parameters given in (a), towards which stationary point does your system converge ?

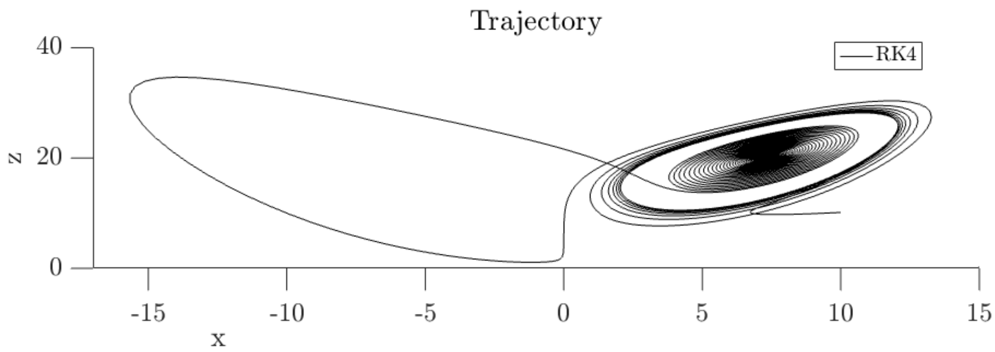


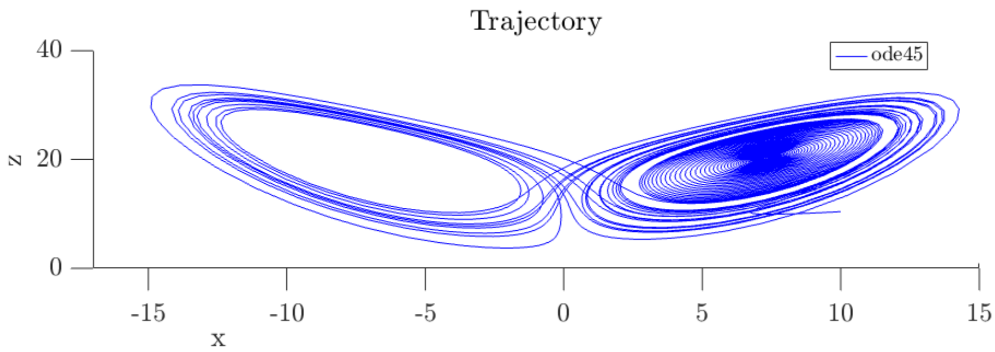
The first equation gives $\dot{X} = 0 = \sigma(Y - X)$. Therefore, for finite viscosity $\sigma \neq 0 \Rightarrow X = Y$. The second equation leads to $\dot{Y} = 0 = -ZX + rX - Y = X(r - 1 - Z)$.

- If $X = 0$, the third equation gives $\dot{Z} = 0 = XY - bZ = bZ$. Since $b \neq 0$, the origin is always a stationary point.
- If $X \neq 0$ then $Z = r - 1$ and the third equation leads to $0 = XY - bZ = X^2 - b(r - 1) \Rightarrow X = \pm\sqrt{b(r-1)}$. Since $b > 0$, these two stationary points, called by Lorenz

$C_\pm$, are $C_\pm = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ and exist only if $r > 1$.
The parameters used give $r - 1 = 9$ and $\sqrt{b(r-1)} \approx 4.8990$. The system does converge towards C_+.

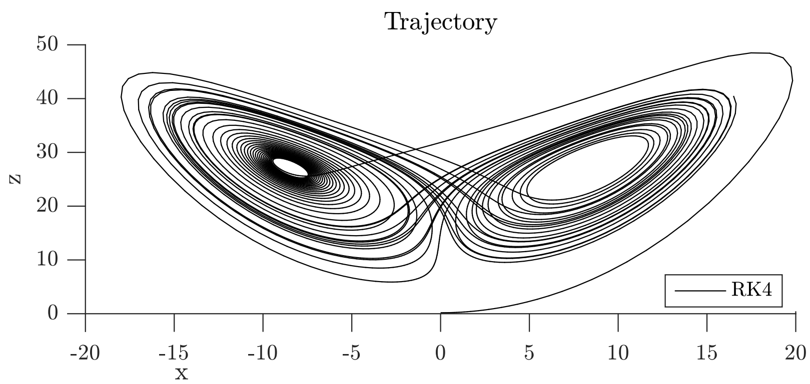
(d) Change to $r = 21$ to study transient chaos. This is where the intermediate state is chaotic and unpredictable (for example, the numerical error produced by different integrators can give completely different results) but, in the end, the system converges to a stationary point. Try $(X_0, Y_0, Z_0) = (10, 0, 10)$ or find other interesting initial conditions.

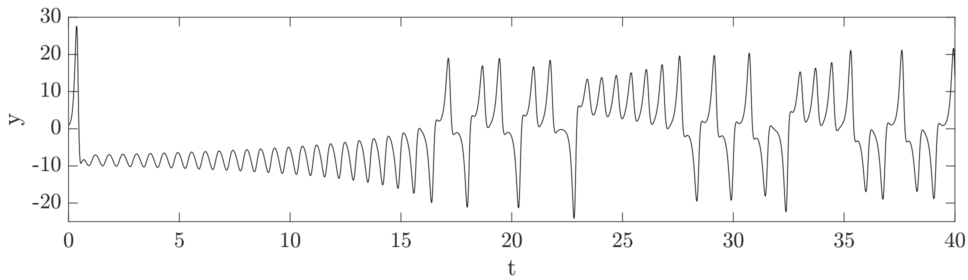




Using the same initial conditions, the chaotic nature of the system gives different results for the Runge-Kutta 4 with $\Delta t = 0.01$ and the `ode45` routine. They switch back and forth looping around the two stationary points, before converging towards one of them.

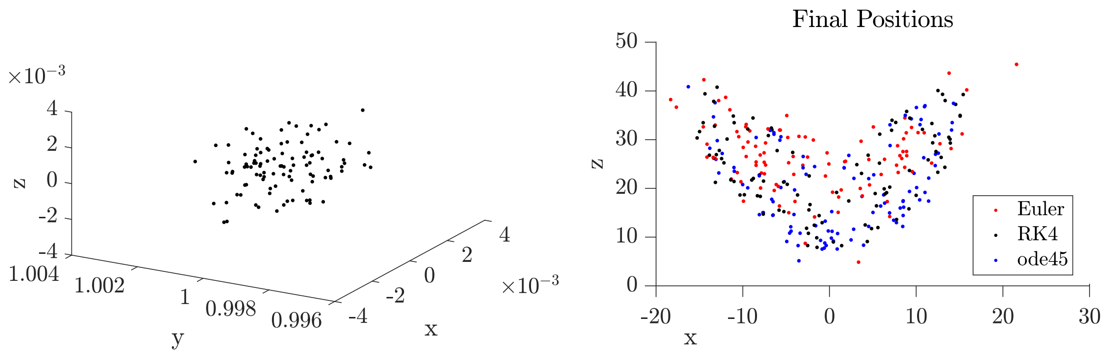
(e) Now for the most famous case studied by Lorenz, use $r = 28$ and $(X_0, Y_0, Z_0) = (0, 1, 0)$ to enter the chaotic regime and observe the trajectories. You can use `comet3` to see the system evolve.





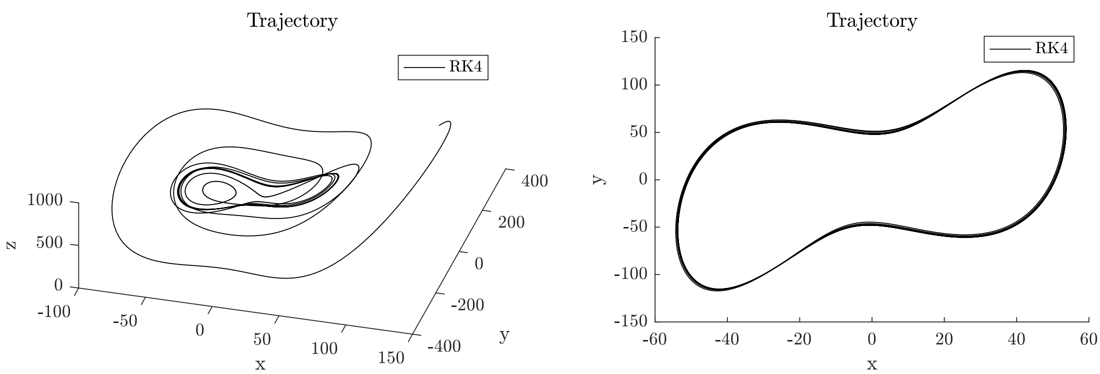
In this chaotic regime the stationary points are no longer stable, the system first goes near C_- but the oscillations grow bigger and bigger until it changes side to oscillate around C_+. The system then evolves chaotically changing between C_+ and C_-, never converging to a stable final state.

(f) Analyse the sensitivity of initial conditions in this chaotic regime. Start by generating a set of $N = 100$ vectors following a Gaussian distribution of mean $(X_0, Y_0, Z_0) = (0, 1, 0)$ and standard deviation $\Delta|\vec{x}_0| = 10^{-3}$. For each of the three variables use `normrnd(X0(i),dX0,[N,1])` to generate a column vector of size `N` with a Gaussian distribution of mean `X0(i)` and standard deviation `dX0`. Integrate using the three methods with parameters $\Delta t = 0.015$ and $t_f = 30$. Plot in phase space only the final positions of these simulations to see how they are distributed.



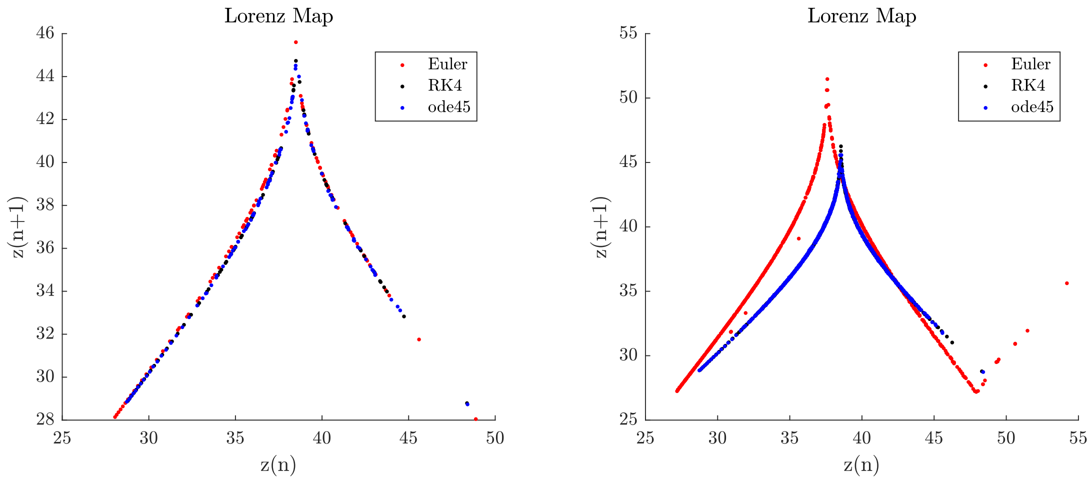
As expected the initial conditions in the first figure are distributed around the $(X_0, Y_0, Z_0) = (0, 1, 0)$ with a standard deviation of 10^{-3}, but the final positions seem uniformly distributed on a large portion of the phase space that is located around the stationary points. Even the different schemes give different results for the same initial conditions. All this is due to the fact that in the chaotic regime, a small perturbation in the initial conditions grows exponentially, making predictions past a certain time scale impossible.

(g) For big $r = 400$, the system can converge to a periodic behavior. Compare the different integrators. For clearer results, try initial conditions near the stable periodic trajectory $(X_0, Y_0, Z_0) = (-54, -59, 485)$.



Even if the initial conditions are not in the neighborhood of the periodic cycle, the trajectory converges towards it. Therefore the periodic cycle is stable. The RK4 scheme and `ode45` function are stable, but Euler is not and very quickly diverges to enormous values.

(h) Is there order in chaos ? In the chaotic regime with $r = 28$, we can try to find some ordered patterns. Looking at the oscillations of variable Z, find the values of all local maxima of Z (Matlab's `findpeaks` will do it in a breeze). Then plot on a 2D graph the pairs (z_n, z_{n+1}), where z_n is the amplitude of the n^{th} peak. Longer simulations give more (z_n, z_{n+1}) pairs, so clearer results.



The Lorenz maps show that there is a relation between the amplitude of a peak and the aplitude of the next. For a fixed number of iterations, using a bigger time step gives more oscillations, therefore a greater number of points, but makes the numerical errors more visible. If the process is carried out for a sufficiently long time and with sufficient precision, one can observe that the pattern is not a single line, but an infinite set of parallel lines (a fractal pattern).

BONUS

2 Derivation of the Lorenz Equations

Following what we did in class, derive the first equation of the Lorenz model i.e. $\frac{\mathrm{d}}{\mathrm{d}\tau}X = \frac{\nu}{K}(Y - X)$. Starting from the Navier-Stokes equation

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\nabla^2\psi - g\varepsilon\frac{\partial\theta}{\partial x} - \nu\nabla^4\psi = 0, \tag{1}$$

imposes a solution of the form

$$\psi = X(t)\alpha_1 \sin(\pi a x/H)\sin(\pi z/H)$$
$$\theta = Y(t)\alpha_2 \cos(\pi a x/H)\sin(\pi z/H) + Z(t)\alpha_3 \sin(2\pi z/H)$$
$$\text{with} \;\; \alpha_1 = \frac{\sqrt{2}(1+a^2)K}{a}, \; \alpha_2 = \frac{\sqrt{2}\Delta T}{\pi r}, \; \alpha_3 = \alpha_2/\sqrt{2} \;\; \text{and} \;\; r = \frac{g\varepsilon H^3 \Delta T a^2}{\nu K \pi^4 (1+a^2)^3}$$

To compute the derivatives, softwares capable of symbolic computation like Mathematica can be of great help. To define a function use command `f[x_,y_]:= a Cos[x] + x f2[y]`. Mathematica does not need know what function `f2` or variable `a` are to do litteral computation. `Derivative[0,3][f][b,c]` can be used for derivatives[1]. This here is the third derivative of `f` in the second variable (which is

1. https ://reference.wolfram.com/language/ref/Derivative.html

y in this case) evaluated at x=b and y=c. At the end make sure to renormalize to the dimensionless time $\tau = \frac{\pi^2(1+a^2)K}{H^2}t$.

One can define :

```
ψ[t_,x_,z_]:=X[t] α1 Sin[πa x/H] Sin[πz/H]
θ[t_,x_,z_]:=Y[t] α2 Cos[πa x/H] Sin[πz/H]+Z[t] α3 Sin[2πz/H]
Lapψ[t_,x_,z_]:=Derivative[0,2,0][ψ][t,x,z]+Derivative[0,0,2][ψ][t,x,z]
```

Asking the software to compute and simplify the left-hand side of equation (1) gives :

```
Simplify[Derivative[1,0,0][Lapψ][t,x,z]
- Derivative[0,0,1][ψ][t,x,z] Derivative[0,1,0][Lapψ][t,x,z]
+ Derivative[0,1,0][ψ][t,x,z] Derivative[0,0,1][Lapψ][t,x,z]
- g ϵ Derivative[0,1,0][θ][t,x,z]
- ν (Derivative[0,2,0][Lapψ][t,x,z] + Derivative[0,0,2][Lapψ][t,x,z]) ]
```

Out=$-\frac{1}{H^4}\pi$ Sin[$\frac{\pi z}{H}$] Sin[$\frac{\pi a x}{H}$] ($H^2(\pi(a^2+1)\alpha_1$ X'[t]-a α_2 g H ε Y[t])+$\pi^3(a^2+1)^2\alpha_1$ ν X[t])

Now it is our turn to equate this to zero, simplify further, and renormalize :

$$\begin{aligned}
H^2\left(\pi\left(a^2+1\right)\alpha_1 X'(t) - a\alpha_2 gH\varepsilon Y(t)\right) + \pi^3\left(a^2+1\right)^2\alpha_1\nu X(t) &= 0\\
H^2\left(\pi\left(a^2+1\right)\alpha_1 X'(t) - a\alpha_2 gH\varepsilon Y(t)\right) &= -\pi^3\left(a^2+1\right)^2\alpha_1\nu X(t)\\
H^2\pi\left(a^2+1\right)\alpha_1 X'(t) &= H^2 a\alpha_2 gH\varepsilon Y(t) - \pi^3\left(a^2+1\right)^2\alpha_1\nu X(t)\\
\frac{H^2}{\pi^2(1+a^2)K}X'(t) &= \frac{a\alpha_2 gH^3\varepsilon}{\alpha_1\pi^3(1+a^2)^2K}Y(t) - \frac{\nu}{K}X(t)\\
\frac{\mathrm{d}t}{\mathrm{d}\tau}X'(t) &= \frac{a\alpha_2 gH^3\varepsilon}{\alpha_1\pi^3(1+a^2)^2K}Y(t) - \frac{\nu}{K}X(t)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{agH^3\varepsilon}{\pi^3(1+a^2)^2K}\frac{\alpha_2}{\alpha_1} &= \frac{agH^3\varepsilon}{\pi^3(1+a^2)^2K}\frac{\sqrt{2}\Delta Ta}{\sqrt{2}(1+a^2)K\pi r}\\
&= \frac{a^2gH^3\varepsilon\Delta T}{\pi^4(1+a^2)^3K^2}\frac{1}{r}\\
&= \frac{a^2gH^3\varepsilon\Delta T}{\pi^4(1+a^2)^3K^2}\frac{\nu K\pi^4(1+a^2)^3}{g\varepsilon H^3\Delta Ta^2}\\
&= \frac{\nu}{K}
\end{aligned}$$

We have indeed,
$\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\nu}{K}Y - \frac{\nu}{K}X$, that is
$\frac{\mathrm{d}X}{\mathrm{d}\tau} = \sigma(Y - X)$.