6 May 2024

Solutions 10 : One-Dimensional Maps (2)

1 Cubic Map

Similarly to the logistic map, a map that appears simple but also exhibits choatic behaviour is the cubic map

$$x_{n+1} = f_a(x_n) \quad \text{with} \quad f_a(x) = ax - x^3$$

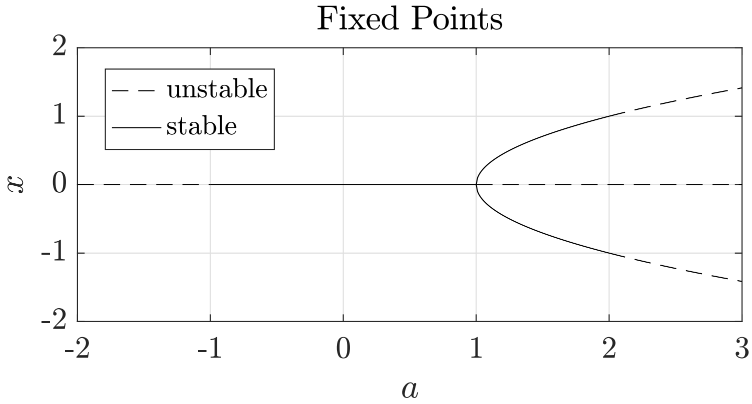
with $a \in \mathbb{R}$.

(a) Determine the fixed points and identify for which values of a they exist.

To find the fixed points we solve $x_* = f_a(x_*) = ax_* - x_*^3$, which is equivalent to $0 = x_*[(a-1) - x_*^2]$. The first solution is $x_* = 0$, the fixed point exists for all a. Then, if $a > 1$ there are two additional fixed points $x_* = \pm\sqrt{a-1}$.

(b) Determine the stability of the fixed points to qualitatively draw a bifurcation diagram. Do you recognise a familiar bifurcation ?

To study the stability of the fixed point, we evaluate the derivative of f_a at the fixed points. Since $f_a'(x) = a - 3x^2$, at the origin $f_a'(x_* = 0) = a$. So the origin is stable for $a \in]-1, 1[$ unstable for $a \in]-\infty, -1[\cup]1, \infty[$. At $a = \pm 1$, further analysis is needed to determine the stability. For the other two fixed points $f_a'(\pm\sqrt{a-1}) = a - 3(a-1) = 3 - 2a$. These fixed points exist only if $a > 1$. They are stable for $1 < a < 2$ since it implies $-1 < 3 - 2a < 1$. They are unstable for $a > 2$ since it implies $3 - 2a < -1$. At $a = 2$, further analysis is needed to determine the stability. We recognise a supercritical pitchfork bifurcation at $a = 1$. There are also flip bifurcations at $a \in \{-1, 2\}$, which are usually associated with period doubling.



BONUS

(c) For $a \leq -1$ prove that the sequence diverges, that is, $\lim_{n\to\infty} |x_n| = +\infty$, $\forall x_0 \neq 0$.
Help : Start by showing that $|x_{n+1}| > |x_n|$.

In general

$$|x_{n+1}| = |f_a(x_n)| = |ax_n - x_n^3| = |x_n| \cdot |a - x_n^2| \geq |x_n| \cdot |1 + x_n^2| \tag{1}$$

The last inequality is motivated by the fact that $a \leq -1$ implies that $a - x_n^2 \leq -1 - x_n^2 < 0$ so $|a - x_n^2| \geq |1 + x_n^2|$. Equation (1) implies that the sequence is monotonically increasing $|x_{n+1}| \geq |x_n|$. Therefore $|1 + x_n^2| \geq |1 + x_0^2|$. By induction,

$$|x_n| \geq |x_{n-1}| \cdot |1 + x_0^2| \geq |x_{n-2}| \cdot |1 + x_0^2|^2 \geq \cdots \geq |x_0| \cdot |1 + x_0^2|^n \tag{2}$$

Since $x_0 \neq 0$ then $|1+x_0^2| > 1$, therefore the sequence $|x_0| \cdot |1+x_0^2|^n$ diverges. Equation (2) implies that $|x_n|$ diverges as well. Note that this proves that the fixed point $x_* = 0$ is unstable for $a \leq -1$ (even at $a = -1$).

(d) Using Mathematica, find the 2-cycles of f_a. Identify for which values of a they exist.

To find the 2-cycles, one needs to find the fixed points of $f_a(f_a(x)) = f_a^2(x)$. Since f_a^2 is a polynomial of order 9, it is preferable to ask Mathematica

```
f[a_,x_]:= a x - x^3
Solve[f[a, f[a, x]] == x, x]
```

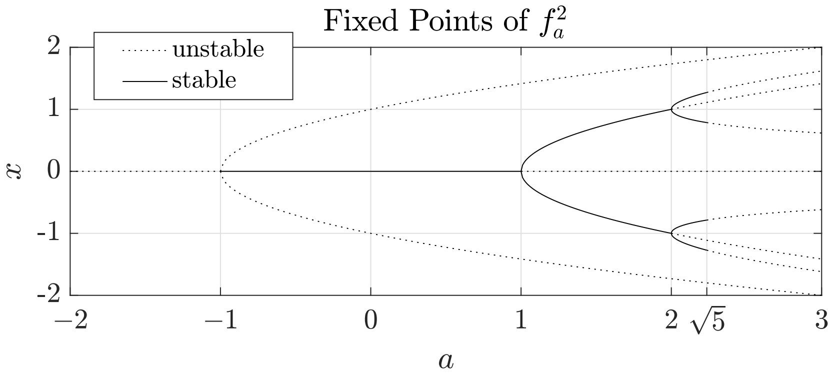
Some of the solutions look familiar, $x = 0$ and $x = \pm\sqrt{a-1}$. It is because fixed points of f_a are also fixed points of f_a^2. The six other fixed points of f_a^2, that lead to proper 2-cycles, are $\pm\sqrt{a+1}$ and $\pm\frac{1}{\sqrt{2}}\sqrt{a \pm \sqrt{a^2-4}}$. Since $f_a(\sqrt{a+1}) = -\sqrt{a+1}$, therefore the pair $\{\sqrt{a+1}, -\sqrt{a+1}\}$ forms a 2-cycle, which exists only for $a > -1$. One can also check that $f_a(\frac{1}{\sqrt{2}}\sqrt{a+\sqrt{a^2-4}}) = \frac{1}{\sqrt{2}}\sqrt{a-\sqrt{a^2-4}}$, therefore the pair $\{\frac{1}{\sqrt{2}}\sqrt{a+\sqrt{a^2-4}}, \frac{1}{\sqrt{2}}\sqrt{a-\sqrt{a^2-4}}\}$ forms a 2-cycle, which exists only for $a > 2$. Indeed this is because $\sqrt{a^2-4}$ is only defined for $a \geq 2$, and $a > \sqrt{a^2-4}$ means that $\sqrt{a \pm \sqrt{a^2-4}}$ does define 2 distinct points for $a > 2$. Similarly the pair $\{-\frac{1}{\sqrt{2}}\sqrt{a+\sqrt{a^2-4}}, -\frac{1}{\sqrt{2}}\sqrt{a-\sqrt{a^2-4}}\}$ forms a 2-cycle, which exists only for $a > 2$.

(e) Again with the help of Mathematica, determine the stability of the 2-cycles and qualitatively draw a bifurcation diagram.

The stability of the fixed points is already known, computed in question (b). To evaluate the stability of the 2-cycles we need to evaluate the derivative $(f_a^2)'$ and the fixed points. It is done rapidly with Mathematica

```
f2[a_,x_] := f[a, f[a, x]]
Derivative[0,1][f2][a, x]
```

We obtain $(f_a^2)'(\pm\sqrt{a+1}) = (3+2a)^2$. Since $a > -1$ implies that $3+2a > 1$ the associated 2-cycle is always unstable. Moving on the next cycle $(f_a^2)'(\frac{1}{\sqrt{2}}\sqrt{a \pm \sqrt{a^2-4}}) = 9-2a^2$. We remember that this limit cycle exists only for $a > 2$. One can check that for $2 < a < \sqrt{5}$ the cycle is stable since $-1 < 9-2a^2 < 1$, and for $\sqrt{5} < a$ the cycle is unstable since $9-2a^2 < -1$. Since f_a is odd, then so is f_a^2, meaning the $(f_a^2)'$ is even. So the 2-cycle $-\frac{1}{\sqrt{2}}\sqrt{a \pm \sqrt{a^2-4}}$ has the same stability properties.



(f) For any $-1 < a \leq 3$, iterate numerically $x_{n+1} = f_a(x_n)$ with $x_0 = \sqrt{a+1} \pm \epsilon$, with $\epsilon = 10^{-2}$. Identify the set of initial conditions for which the iterations diverge, i.e. $\lim_{n\to\infty} |x_n| = \infty$, and explain the numerical results.

The simulations are started just inside and just outside of the unstable 2-cycle. It is observed, and can be proved analytically that initial conditions with $|x_0| > \sqrt{a+1}$ diverge. Similarly, simulations with $|x_0| \leq \sqrt{a+1}$ and $-1 < a \leq 3$ stay bounded for all $n \in \mathbb{N}$.

BONUS

(g) Suppose a sequence with initial condition x_0 such that $|x_0| < \sqrt{a+1}$. For $-1 < a \leq 3$, since $|f_a(x)| \leq \sqrt{a+1}$ for all $|x| \leq \sqrt{a+1}$, the sequence x_n is bounded from above by $\sqrt{a+1}$, i.e. $|x_n| < \sqrt{a+1}$ for all n. On the other hand, for $a > 3$ a sequence with $|x_0| < \sqrt{a+1}$ can diverge. Explain why this happens.
Hint : Start by identifying the local maximum of f_a (when $a > 0$).

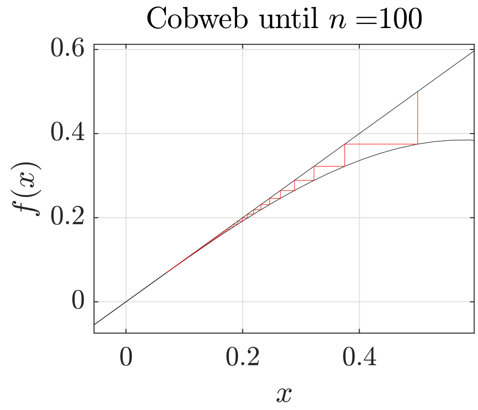
To find the local extrema of f_a we look where the derivative vanishes. It is $f_a'(x) = a - 3x^2$, which vanishes at $x = \pm\sqrt{a/3}$, when $a > 0$. The local maximum is at $x_{\max}(a) = \sqrt{a/3}$. We note that $x_{\max}(a) < \sqrt{a+1}$, the maximum is in the bounded domain. If for some $|x| < \sqrt{a+1}$, we have $|f_a(x)| > \sqrt{a+1}$, then the trajectories that pass through that x will escape the bounded region and diverge to infinity. Therefore trajectories will be able to escape the bounded domain $[-\sqrt{a+1}, \sqrt{a+1}]$ if

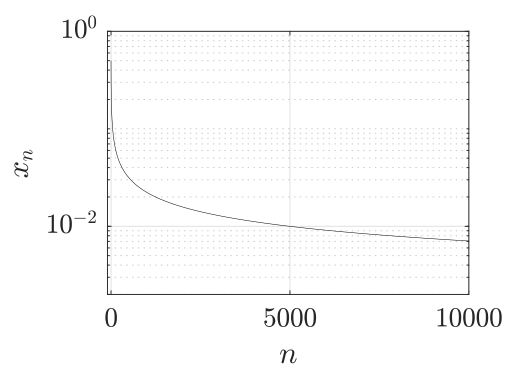
$$\begin{aligned} |f_a(x_{\max}(a))| &> \sqrt{a+1} \\ |a\sqrt{a/3} - \sqrt{a/3}^3| &> \sqrt{a+1} \\ (2a/3)\sqrt{a/3} &> \sqrt{a+1} \\ 4a^3/27 &> a+1 \end{aligned}$$

To solve this make a plot of the polynomial $4a^3/27 - a - 1$, or use command `Reduce[4a^3/27>a+1,a]` in Mathematica. The polynomial has a double root at $a = -3/2$, around which it is negative, and a root at $a = 3$ where the polynomial becomes positive. We want $4a^3/27 - a - 1 > 0$, which is therefore when $a > 3$. Trajectories diverge for $a > 3$ even if $|x_0| < \sqrt{a+1}$.

(h) For $a = 1$ numerically determine the stability of the fixed point. Discuss the convergence speed.

In question (b) we saw that $f_1'(0) = 1$, so the stability was unknown. From the supercritical pitchfork in the bifurcation diagram, it can be concluded that $x_* = 0$ is a stable fixed point with slow convergence. The convergence can be shown numerically. A logarithmic scale shows that the convergence is slow as the slope keeps getting flatter.





Our map experiences period doubling as a is increased from 1 to $a_{\text{chaos}} \approx 2.30228346$. For each 2^r-cycle, there is a value $a = A_r$ at which a point of the 2^r-cycle coincides with $x_{\max}(A_r)$. As $r \to \infty$, the value of $\delta_r = \frac{A_{r+1} - A_r}{A_{r+2} - A_{r+1}}$ should converge to $\delta \approx 4.669$, one of the Feigenbaum constants. Let's numerically compute the first digits of this constant. Here we suggest how to do it.

(i) On Matlab, implement code the function `rel_pos_xnear`. The function takes as input a (with $1 \le a \le a_{\text{chaos}}$) and iterates the map using as initial condition $x_0 = 0.5$. You should expect the iteration to converge to a 2^r-cycle. Then, the function finds x_{near}, the point on the cycle that is nearest to $x_{\max}$. The function returns the "relative position" between x_{near} and $x_{\max}(a)$, which we define as $+1$ if $x_{\text{near}} > x_{\max}$, 0 if $x_{\text{near}} = x_{\max}$ and -1 if $x_{\text{near}} < x_{\max}$. You can test the function with $a = 2.1$ where $x_{\text{near}} > x_{\max}$ and with $a = 2.2$ where $x_{\text{near}} < x_{\max}$.

See the `rel_pos_xnear.m` file for the solution.

(j) Still on Matlab, implement the function `dichotomy`. This function takes as input a lower and upper bound of a, a_{inf} and a_{sup}, in between which we suppose there is a single A_r. By definition, at $a = A_r$ the relative position between $x_{\max}$ and x_{near} changes. The function uses this to perform the dichotomy algorithm to find A_r up to a given precision. It then returns A_r as output. Test the function with input $a_{\text{inf}} = 2.1$ and $a_{\text{sup}} = 2.2$. It should return $A_1 \approx 2.12$ (hopefully with more significant figures).

See the `dichotomy.m` file for the solution.

(k) On Matlab, write a script that successively finds the values A_r. To do so, divide the interval $[1, a_{\text{chaos}}]$ linearly into approximatly 50 values a_i. Starting at the smallest a_1, determine the relative position between x_{near} and $x_{\max}$, then increase a until a_j is reached, where this relative position changes. Apply the dichotomy algorithm to precisely obtain the $A_r \in]a_{j-1}, a_j[$. Then repeat the algorithm replacing $[1, a_{\text{chaos}}]$ with $[a_j, a_{\text{chaos}}]$.

See the beginning of the `Feigenbaum_constants.m` file for the solution.

(l) Use the obtained values of A_r to compute the values of δ_r. Try to evaluate at least δ_5, which provides an estimate of δ with an error smaller than 10^{-4}.
<u>*Hint*</u> : The nature of the period doubling makes it difficult to compute succesive A_r. Note that, to compute A_r, the function `rel_pos_xnear` should be able to find x_{near} in a 2^r-cycle and this requires that the iterations have sufficiently converged to this cycle. If `rel_pos_xnear` works perfectly, then the error on A_r is set by `dichotomy`. Since A_{13} has the same first significant figures as the finite number of digits we have given for a_{chaos} this method cannot go further.

It is done in the `Feigenbaum_constants.m` file. A good first choice of parameters are : in `dichotomy.m` `p=1e-10` of precision, in `rel_pos_xnear.m` a length of simulation of `N=1000` and look only at the last `Nlast=2^7` converged points. These low end parameters make the program run fast and obtains the `delta_5 = 4.66904`. To go up to a decent A_{11} one can use `p=1e-14`, `N=15000` and `Nlast=2^11`.

At $a = A_r$, one point of the 2^r-cycle coincides with $x_{\max}(A_r)$. The distance with the closest other point of the cycle is d_r. The ratio $\alpha_r = \frac{d_r}{d_{r+1}}$ convergences to $\alpha \approx -2.5029$, the second Feigenbaum constant, as $r \to \infty$. Again, let us obtain it numerically.

(m) Implement the function `dist_second_nearest`. It takes as input a value A_r as well as the $r \ge 1$ it is associated with, and returns d_r. To do so, it launches a simulation from $x_0 = 0.5$, which should converge to the 2^r-cycle with $x_n > 0$. Only keep the last 2^r points of the simulation, in order to have exactly one cycle. Then d_r is the difference between $x_{\max}(A_r)$ and the second closest point to $x_{\max}(A_r)$. Use that function to compute the first iterations α_r.

See the `dist_second_nearest.m` file for the function. The end of file `Feigenbaum_constants.m` computes α_r.