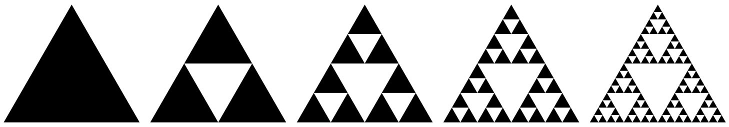
20 May 2025

Problem Set 12 : Fractals

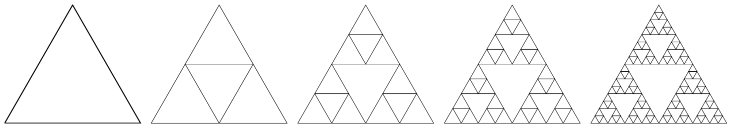
1 Similarity Dimension

The Sierpinski triangle fractal is the object towards which the following iterations converge



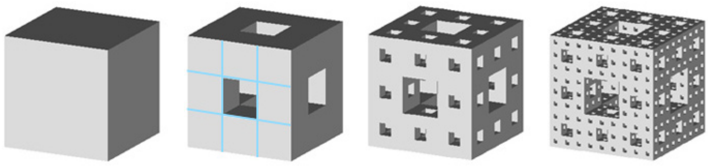
(a) The side of the initial triangle has a length of 1. Compute the area of the Sierpinski triangle and deduce that its dimension is smaller than 2.

(b) Now show that the "length" of the Sierpinski triangle is infinite, meaning that its dimension is greater than 1. To do so, notice how from one iteration to the next, the border of the shape stays inside the Sierpinski triangle and increases in size.



(c) Compute the similarity dimension of the Sierpinski triangle.

The Menger sponge fractal is the fractal towards which the following iterations converge

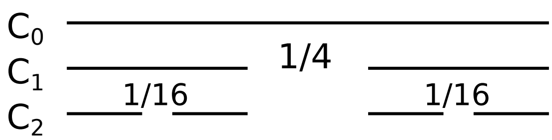


We start with a cube, split it into 27 smaller cubes, remove the ones that form the "central cross", then repeat for all the smaller cubes left.

(d) Compute the similarity dimension of the Menger sponge.

2 Fat Fractals

Fat fractals are fractals with a non-zero measure. Remember that for the Cantor middle set, S_0 starts with the interval $[0, 1]$ and then, from S_n to S_{n+1}, an open interval of size $1/3^{n+1}$ is removed from the middle of each sub-interval in S_n. The Cantor middle set is the limit S_∞. An example of a fat fractal, similar to the Cantor middle set, starts from C_0, which is the interval $[0, 1]$, and then, from C_n to C_{n+1}, intervals of size $1/4^{n+1}$ are removed.



(a) Briefly explain why this fractal is a topological Cantor set.

(b) Compute the measure of the fractal, to prove it is indeed fat.

(c) What is therefore the dimension of this fractal ?

Fat fractals are linked to an important aspect of the logistic map. Farmer numerically proved in 1985 that the set of parameter values of a that lead to chaos is a fat fractal. He found that if a value of a is randomly chosen in the interval $[a_c \approx 3.57, 4]$ the resulting map is chaotic 89% of the time.

(d) On Matlab, generate a random, uniformly distributed, parameter value a inside $[a_c, 4]$, then iterate the map and determine whether the behaviour is chaotic or periodic. Average over multiple examples to compute the fraction at which chaos occurs.

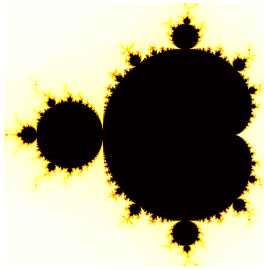
Help : To generate a uniformly distributed number use command `rand`. To see if a map is periodic, iterate for long enough and apply the discrete Fourier transform. To do so use commands

```
DFT = fft(x);
DFT = abs(DFT/N);
DFT = DFT(2:N/2);
```

where `x` is the vector of size `N` containing the map. The absolute value is taken to get the amplitudes of each frequency. The third line is to get the single-sided spectrum. If you want to plot this, the amplitudes of `DFT` correspond to frequencies `freq=(1:N/2-1)/N`. You can then say that the map is periodic if the maximum value of `DFT` is above a certain threshold.

3 Mandelbrot set

We consider the following sequence :



$$f : z_{n+1} = z_n^2 + c$$

where $z, c \in \mathbb{C}$, $c = x + iy$, and $z_0 = 0$. The Mandelbrot set is the set of complex numbers c such that f is bounded when $n \to \infty$. This set has been defined, when discovered, as "the most complex object in Mathematics ". It is one of the best known examples of mathematical beauty. The boundary of this set in the complex space is a fractal.

(a) On Matlab, write a script that shows graphically the Mandelbrot set in the complex plane. Define a rectangle of 2000×2000 points in the (x, y) plane. Consider $-1.5 \leq x \leq 0.5$ and $-1.0 \leq y \leq 1.0$. Define a maximum absolute value z_{max} (e.g. 4) such that the sequence f is considered as bounded if :

$$\lim_{n \to \infty} |z_{n+1}| < z_{max}$$

Now iterate the sequence f on the defined space, starting from $z_0 = (0, 0)$. For each iteration, assign a different value k (for example the number of the iteration itself) to the set of points where f is bounded, then plot k as a function of x and y. Find an appropriate number of iterations in order to converge to the Mandelbrot set.

(b) Discuss the self-similarity of the boundary of the Mandelbrot set. You can refine the resolution of your box, in order to see finer structures.

(c) Estimate the fractal dimension of the boundary of the Mandelbrot set with the box counting method. Use the Matlab function `contour`, to trace a contour at a specific value of k. For sufficiently high k, the points found by the `contour` command approximate well the boundary of the Mandelbrot set. In order to calculate the fractal dimension, evaluate the number of points found by `contour` as function of the number of the (x, y) grid points.

4 Lorenz Attractor Dimension

In this exercise, we evaluate the correlation dimension of the Lorenz attractor. The correlation dimension is a measure of the fractal structure of an attractor. In particular, consider the set of points $\{\vec{x}_i, i = 1, 2, ...N\}$ on an attractor, obtained from a sequence with a fixed time increment $\tau : \vec{x}_i \equiv \vec{x}(t + i\tau)$. In *Grassberger and Procaccia (1983)*, the correlation integral is defined as

$$C(l) = \lim_{N \to \infty} \frac{1}{N^2} \times \{number\ of\ pairs\ (i, j) : |\vec{x}_i - \vec{x}_j| < l\}\,, \tag{1}$$

where $l > 0$. One finds that, for small values of l,

$$C(l) \sim l^{\nu}\,, \tag{2}$$

where the exponent ν is the correlation dimension.

(a) On Matlab, write a program that calculates the correlation dimension of the Lorenz attractor. First integrate the Lorenz system with parameters $\sigma = 10$ and $b = 8/3$. As a first step, choose $r = 28$, a value that leads chaotic evolution. As in *Grassberger and Procaccia (1983)*, use $N = 15000$ points in the integration of the Lorenz system. Count all the pairs of points (i, j)

that satisfy the condition $|\vec{x}_i - \vec{x}_j| < l$, for several values of l, with l in the interval $[l_{\text{min}}, l_{\text{max}}]$, with l_{max} such that $C(l_{\text{max}}) = 1$, i.e. all pairs of points have a relative distance lower than l_{max}, and l_{min} given by the minimum distance between two points.

(b) Study how the correlation dimension varies with the r parameter.