*13May 2024*

# Problem Set 11 : Renormalisation

## 1 Renormalised Plots

The goal of this exercise is to visualise that the functions $f^{2^r}(A_r, x)$ can be renormalised to look very similar. Consider the logistic equation $f(a, x) = ax(1 - x)$.

(a) First consider $a = a_c \approx 3.5699$ (at $a = a_c$ the system becomes chaotic). Plot four graphs $y = f^{2^r}(a_c, x)$, on interval $x \in [0, 1]$, with $r \in \{0, 1, 2, 3\}$.

(b) Add the diagonal $y = x$ on the plots. The intersections where $x = f^n(a_c, x)$ correspond to fixed points, which are all unstable since $a = a_c$. A fixed point of $f^n$ corresponds to a point of an $m$-cycle, where $m$ is a divisor of $n$. On the graph of $f^2$, identify the fixed points and the period of the cycle associated. Try to identify the $4$-cycle using the graph of $f^4$.

(c) Now animate the plots of $f^{2^r}$, $r \in \{0, 1, 2, 3\}$ by varying $a$. For all four graphs consider the range $0 \le a \le a_c$. Identify the flip bifurcations and period doubling.

(d) The graphs $f^{2^r}$ all look the same near their associated $A_r$, close to the maximum at $x = 0.5$. To visualise it, animate again the graphs, this time for $f^{2^0}$ use the range $1 \le a \le A_0$, and in general $A_{r-1} \le a \le A_r$ for $f^{2^r}$. Also magnify the plots as $r$ increases by using a plot range centered at $(x, y) = (0.5, 0.5)$ that is a square of side $1/\alpha^r$.
<u>*Help*</u> : Use $A_0 = 2$, $A_1 = 3.2361$, $A_2 = 3.4986$, $A_3 = 3.5546$, $A_4 = 3.5667$.

## 2 Universal Constant

Consider the function $f(a, x) = a - (x - 2)^2$ which has a quadratic maximum.

(a) Find $x = x_m$, the location of the maximum of $f$. Then, find $h(a, x)$, the translation of $f$ with maximum at $x = 0$.

(b) Numerically plot the orbit diagram of $h$ for $-1/4 \le a \le 2$, using $x_0 = 0$. Graphically determine $A_r$, with $r = \{0, 1\}$, the values of $a$ at which the $2^r$-cycle has a point that coincides with the maximum of $h$.

(c) Identify the explicit expressions of the functions $g_{0r}(x) = \alpha^r h^{2^r}(A_r, \frac{x}{\alpha^r})$, for $r = \{0, 1\}$.

(d) From what we have learned in class, the functions $g_{0r}$, which are renormalised using the universal constant $\alpha$, should resemble each other near $x = 0$, and should converge to a function $g_0$ as $r \to \infty$. Keeping the lowest-order terms, what should be the value of $\alpha$ so that $g_{00} \approx g_{01}$ near $x = 0$ ? How does this compare with the Feigenbaum constant ?

(e) Using Mathematica, show that $A_2$ is the root of a $8^{\text{th}}$ order polynomial, and numerically find it.

(f) Using Mathematica, express the lowest order term of $g_{02}(x)$, which is proportional to $x^2$, as a function of $\alpha$ and $A_2$. Then, use the numerical value of $A_2$ to compute the value of $\alpha$ that would make the lowest-order terms of $g_{01}$ and $g_{02}$ equal.

## 3 Quartic Chaos

Up to now we have seen the path to chaos around the quadratic maximum of a map $f$. The renormalisation was carried out by using the universal constants $\alpha$ and $\delta$. These universal constants are different if we study a map presenting a quartic maximum. Similarly to the quadratic maximum, for a function $f$ with a quartic maximum at $x = 0$ we can renormalise $g_{qr}(x) = \alpha^r f^{2^r}(A_{r+q}, \frac{x}{\alpha^r})$. Then, $g_q(x) = \lim_{r \to \infty} g_{qr}(x)$ and the universal function $g(x) = \lim_{q \to \infty} g_q(x)$ has the property $g(x) = \alpha g(g(\frac{x}{\alpha}))$.

(a) We expect $g(x)$ to behave like $g_{00}(x) = f(A_0, x)$ near $x = 0$. Suppose therefore that $g(x) = 1 + bx^4$ and, neglecting the higher-order terms, use the equation $g(x) = \alpha g(g(\frac{x}{\alpha}))$ to determine an approximate value of $b$ and $\alpha$.

(b) Numerically plot the orbit diagram of $f(a,x) = a - x^4$ for $-0.25 \leq a \leq 1.26$ using $x_0 = 0$.

(c) Now, compute an approximate value of $\delta$ from $A_0$, $A_1$ and $A_2$.
*Help* : Use the command `NSolve` in Mathematica and do not go any further than $A_2$ as computations get very complicated.