6 May 2024

Problem Set 10 : One-Dimensional Maps (2)

1 Cubic Map

Similarly to the logistic map, a map that appears simple but also exhibits choatic behaviour is the cubic map

$$x_{n+1} = f_a(x_n) \quad \text{with} \quad f_a(x) = ax - x^3$$

with $a \in \mathbb{R}$.

(a) Determine the fixed points and identify for which values of a they exist.

(b) Determine the stability of the fixed points to qualitatively draw a bifurcation diagram. Do you recognise a familiar bifurcation ?

BONUS

(c) For $a \leq -1$ prove that the sequence diverges, that is, $\lim_{n\to\infty} |x_n| = +\infty$, $\forall x_0 \neq 0$.
Help : Start by showing that $|x_{n+1}| > |x_n|$.

(d) Using Mathematica, find the 2-cycles of f_a. Identify for which values of a they exist.

(e) Again with the help of Mathematica, determine the stability of the 2-cycles and qualitatively draw a bifurcation diagram.

(f) For any $-1 < a \leq 3$, iterate numerically $x_{n+1} = f_a(x_n)$ with $x_0 = \sqrt{a+1} \pm \epsilon$, with $\epsilon = 10^{-2}$. Identify the set of initial conditions for which the iterations diverge, i.e. $\lim_{n\to\infty} |x_n| = \infty$, and explain the numerical results.

BONUS

(g) Suppose a sequence with initial condition x_0 such that $|x_0| < \sqrt{a+1}$. For $-1 < a \leq 3$, since $|f_a(x)| \leq \sqrt{a+1}$ for all $|x| \leq \sqrt{a+1}$, the sequence x_n is bounded from above by $\sqrt{a+1}$, i.e. $|x_n| < \sqrt{a+1}$ for all n. On the other hand, for $a > 3$ a sequence with $|x_0| < \sqrt{a+1}$ can diverge. Explain why this happens.
Hint : Start by identifying the local maximum of f_a (when $a > 0$).

(h) For $a = 1$ numerically determine the stability of the fixed point. Discuss the convergence speed.

Our map experiences period doubling as a is increased from 1 to $a_{\text{chaos}} \approx 2.30228346$. For each 2^r-cycle, there is a value $a = A_r$ at which a point of the 2^r-cycle coincides with $x_{\max}(A_r)$. As $r \to \infty$, the value of $\delta_r = \frac{A_{r+1} - A_r}{A_{r+2} - A_{r+1}}$ should converge to $\delta \approx 4.669$, one of the Feigenbaum constants. Let's numerically compute the first digits of this constant. Here we suggest how to do it.

(i) On Matlab, implement code the function `rel_pos_xnear`. The function takes as input a (with $1 \leq a \leq a_{\text{chaos}}$) and iterates the map using as initial condition $x_0 = 0.5$. You should expect the iteration to converge to a 2^r-cycle. Then, the function finds x_{near}, the point on the cycle that is nearest to $x_{\max}$. The function returns the "relative position" between x_{near} and $x_{\max}(a)$, which we define as $+1$ if $x_{\text{near}} > x_{\max}$, 0 if $x_{\text{near}} = x_{\max}$ and -1 if $x_{\text{near}} < x_{\max}$. You can test the function with $a = 2.1$ where $x_{\text{near}} > x_{\max}$ and with $a = 2.2$ where $x_{\text{near}} < x_{\max}$.

(j) Still on Matlab, implement the function `dichotomy`. This function takes as input a lower and upper bound of a, $a_{\inf}$ and $a_{\sup}$, in between which we suppose there is a single A_r. By definition, at $a = A_r$ the relative position between $x_{\max}$ and x_{near} changes. The function uses this to perform the dichotomy algorithm to find A_r up to a given precision. It then returns A_r as output. Test the function with input $a_{\inf} = 2.1$ and $a_{\sup} = 2.2$. It should return $A_1 \approx 2.12$ (hopefully with more significant figures).

(k) On Matlab, write a script that successively finds the values A_r. To do so, divide the interval $[1, a_{\text{chaos}}]$ linearly into approximatly 50 values a_i. Starting at the smallest a_1, determine the relative position between x_{near} and x_{max}, then increase a until a_j is reached, where this relative position changes. Apply the dichotomy algorithm to precisely obtain the $A_r \in]a_{j-1}, a_j[$. Then repeat the algorithm replacing $[1, a_{\text{chaos}}]$ with $[a_j, a_{\text{chaos}}]$.

(l) Use the obtained values of A_r to compute the values of δ_r. Try to evaluate at least δ_5, which provides an estimate of δ with an error smaller than 10^{-4}.
Hint : The nature of the period doubling makes it difficult to compute succesive A_r. Note that, to compute A_r, the function `rel_pos_xnear` should be able to find x_{near} in a 2^r-cycle and this requires that the iterations have sufficiently converged to this cycle. If `rel_pos_xnear` works perfectly, then the error on A_r is set by `dichotomy`. Since A_{13} has the same first significant figures as the finite number of digits we have given for a_{chaos} this method cannot go further.

At $a = A_r$, one point of the 2^r-cycle coincides with $x_{\text{max}}(A_r)$. The distance with the closest other point of the cycle is d_r. The ratio $\alpha_r = \frac{d_r}{d_{r+1}}$ convergences to $\alpha \approx -2.5029$, the second Feigenbaum constant, as $r \to \infty$. Again, let us obtain it numerically.

(m) Implement the function `dist_second_nearest`. It takes as input a value A_r as well as the $r \geq 1$ it is associated with, and returns d_r. To do so, it launches a simulation from $x_0 = 0.5$, which should converge to the 2^r-cycle with $x_n > 0$. Only keep the last 2^r points of the simulation, in order to have exactly one cycle. Then d_r is the difference between $x_{\text{max}}(A_r)$ and the second closest point to $x_{\text{max}}(A_r)$. Use that function to compute the first iterations α_r.