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Problem Set 9 : One-Dimensional Maps

1 Superstable Cycle

(a) Suppose that a well-behaved function f gives rise to a difference system $x_{n+1} = f(x_n)$ that has a p-cycle $\{x_1, \ldots, x_p\}$. The value of

$$\frac{\mathrm{d}}{\mathrm{d}x} f(f(\cdots f(x)))|_{x=x_i} = (f^p)'(x_i)$$

determines the stability of the cycle. Show that $(f^p)'(x_i) = \prod_{j=1}^{p} f'(x_j)$ and therefore it is the same at all points, $1 \leq i \leq p$, of the p-cycle.

(b) A cycle is said superstable if $(f^p)'(x_i) = 0$. Find the cubic equation that a must satisfy so that $x_{n+1} = f(x_n) = 1 - ax_n^2$ has a superstable 3-cycle.

(c) To visualise the superstable cycle found in (b), create a program. Start by computing a few iterations $x_{n+1} = f(x_n)$ and plot x_n as a function of n. Choose the value of a corresponding to the superstable cycle and check that iterations converge to the superstable 3-cycle.

BONUS

(d) Plot the cobweb diagram. This shows both the $y = f(x)$ function and the $y = x$ diagonal. On top of that, for each x_n, the vertical line from $y = x_n$ to $y = f(x_n)$ and the horizontal from $y = f(x_n)$ back to $y = x_n$ are drawn.

(e) A sequence of one in three iterations, e.g. x_{3k} with $k \in \mathbb{N}$, converges to one of the points on the limit cycle. Make a plot of the distance, in absolute value, between such a sequence and the point on the limit cycle towards which it converges. Describe the superstable convergence.

Suppose we want to solve the equation $g(x) = 0$. Newton's method iteratively approximates the roots by evolving the map $x_{n+1} = f(x_n)$ with $f(x) = x - \frac{g(x)}{g'(x)}$, with x_0 an initial guess of the solution.

2 Newton's Method

(a) Prove that under some condition (which you must specify) the fixed points of the Newton map $f(x)$ are the zeros of $g(x)$.

(b) x_* is a root of g and a fixed point of f. What are the stability properties of x_* ?

(c) Numerically implement Newton's method to solve $0 = 1 - x(1-x)^2$ (we are looking for the solution in the interval $[1, 2]$).

(d) In this question, we establish the convergence properties of the Newton method. Prove that, assuming $g'(x)$ always different from zero, and that $g''(x)$ is continuous, the convergence of the Newton's method is quadratic. *Hint* : Start by Taylor-expanding the $g(x)$ near the root x_*, using an explicit form of the second-order remainder.

(e) Let us explore the behaviour when $g'(x^*) = 0$. Explore the $g(x) = x^2$ and $g(x) = x^{1/3}$. What happens in these cases ?

3 Bifurcation Diagram

The most studied difference equation is the logistic equation $x_{n+1} = f_a(x_n) = ax_n(1 - x_n)$. What a complicated behaviour it shows, as a is varied from 0 to 4, despite its very simple form !

(a) Make a program that generates the orbit diagram. Discretise the parameter a on a set of values a_i. For each a_i, iterate the map. After a certain number of iterations, the system converges to a fixed point, a limit cycle, or just exhibits chaotic behaviour. Plot the iterations, after the initial transient, on the vertical line $a = a_i$.

(b) Plot the graphs $(a, f_a^n(1/2))$, for $n \in \{1, 2, 3, 4, 5\}$, on the interval $a \in [3.4, 4]$. Notice how these graphs correspond to the denser regions of the orbit diagram. Try to qualitatively explain why.

(c) The point $(A, X) \approx (3.7, 0.7)$ is particularly dense on the numerical bifurcation diagram. Taking advantage of the observations in (b), use Mathematica to obtain its analytical coordinates.