*15 April 2025*

# Problem Set 8 : Chaos

## 1 The Rössler System

A set of differential equations close to the Lorenz system was studied by the German biochemist Otto Rössler in 1976 to model the dynamics of chemical reactions

$$\begin{cases} \dot{x}_1 = -x_2 - x_3 \\ \dot{x}_2 = x_1 + a x_2 \\ \dot{x}_3 = b + x_3(x_1 - c) \end{cases}$$

(a) In Matlab, adapt the code used in the previous weeks for the Lorenz system to integrate and plot the trajectories in the Rössler system using again the forward Euler and Runge-Kutta $4^{\mathrm{th}}$ order schemes, as well as Matlab's `ode45`. Run a simulation with $a = 1/4$, $b = 1$ and $c = 3/2$ and to verify that the system starting at $\vec{x}_0 = (1, 0, 0)$ converges to a fixed point (using `dt=0.01` should be sufficient).
*BONUS* : adapt the file `compare_dt.m` to verify that the orders of convergence are the expected ones.

(b) Identify the nullclines and the fixed points, carefully treating all cases of possible values of the parameters $a$, $b$ and $c$. For the rest of the exercise, consider only the $a \neq 0$ case.

(c) For given parameters $a$ and $b$, what are the fixed points in the limit of $c \to \infty$ ?

(d) Identify the Jacobian matrix associated with the linearised system and its characteristic polynomial.

(e) At the smallest positive value $c = c_{\mathrm{lim}}$ at which fixed points exist, what is the characteristic polynomial ? Evaluate the eigenvalues and eigenvectors in the case of $a = 1/4$ and $b = 1$ (feel free to use Mathematica's `Eigenvalue` and `Eigenvector`). You should find a real eigenvalue and two complex conjugate eigenvalues.

(f) Still with $a = 1/4$, $b = 1$ and $c = c_{\mathrm{lim}}$, compare the numerical solutions to the predictions that linearisation makes in the neighborhood of the fixed point. For the real eigenvalue, the associated eigenspace is a line, passing through the fixed point. Run numerical simulations with initial conditions in the eigenspace, on both sides of the fixed point. Plot and discuss the trajectories. What could be the bifurcation at $c = c_{\mathrm{lim}}$ ?
*Help* : Use initial conditions located at a distance of approximately $0.1$ from the fixed point. Run the simulation for a time of approximately $100$.

(g) Again with $a = 1/4$, $b = 1$ and $c = c_{\mathrm{lim}}$, verify that the trajectories in the manifold identified by the two complex conjugated eigenvectors are stable spirals, checking that linearisation correctly predicts the behaviour of the trajectories. To do so, look for the eigenvector generating the plane in the following way. Considering the two complex conjugated eigenvectors, $\mathbf{v}$ and $\bar{\mathbf{v}}$, take the plane generators via $\mathbf{u}_1 = Re[\mathbf{v}]$ and $\mathbf{u}_2 = Im[\mathbf{v}]$. The orthogonal direction is given by $\mathbf{u}_1 \times \mathbf{u}_2$. Orthonormalising $\mathbf{u}_1$ and $\mathbf{u}_2$ provides a real basis spanning the spiral plane. Then, project the trajectory on this new basis, and plot separately the spiral plane projected and the normal projection. *Help* : Start the trajectory at a distance of approximately $0.1$ from the fixed point. You can use the Matlab function `orth` to get an orthonormal basis of the plane, and `cross` to get the normal direction of the plane.

(h) Still using $a = 1/4$ and $b = 1$, we are interested in the eigenvalues of the linearised system about the fixed points as $c$ is varied. On Matlab, start from $c = c_{\mathrm{lim}}$. Then, increase $c$. At each value of $c$ compute the coordinates of the fixed points and find the eigenvalues of the Jacobian matrix at the fixed point. Plot the eigenvalues as $c$ is varied.

*Help* : At each iteration of $c$, the `roots` command may not give the eigenvalues in the same order, meaning you may need to sort them (possibly by comparing their imaginary parts).

(i) Observe the dependence on $c$ of the real eigenvalues of the Jacobian matrix at the fixed points, near $c = c_{\mathrm{lim}}$. Does this confirm the bifurcation you had predicted in (f) ?

(j) Observe the complex conjugate eigenvalues of one of the fixed points, what type of bifurcation can we expect at $c = 2$ ?

(k) Discuss stability properties of the fixed points as $c \to \infty$.

(l) Again with $a = 1/4$ and $b = 1$, using $c \in \{3, 4, 4.83, 4.9\}$ consider a few trajectories, and determine the long-term behaviour of the system (convergence to a stable fixed point, limit cycle, strange attractor, etc. . .). Then, plot the Poincaré map of the system in its final state. The Poincaré map is a 2D plot of the intersection between the trajectory and a given plane. We suggest to choose the $x_1 = 0$ plane.