8 April 2025

Problem Set 7 : Bifurcations in Two Dimensions

1 Dulac's Criterion

In biology, a simple competitive version of the logistic model is

$$\begin{cases} \dot{x}_1 = r_1 x_1 \Big(1 - \dfrac{x_1}{N_1}\Big) - b_1 x_1 x_2 \\ \dot{x}_2 = r_2 x_2 \Big(1 - \dfrac{x_2}{N_2}\Big) - b_2 x_1 x_2 \end{cases}$$

(a) What should be the sign of the variables x_1, x_2 and parameters r_1, r_2, N_1, N_2, b_1 and b_2. Explain their physical meaning.

(b) Show that there is no limit cycle for $x_1, x_2 > 0$ using the weighting function $g = \frac{1}{x_1 x_2}$.

2 Hopf Bifurcation

Consider the predator-prey model

$$\begin{cases} \dot{x}_1 = x_1 \Big(b - x_1 - \dfrac{x_2}{1 + x_1}\Big) \\ \dot{x}_2 = x_2 \Big(\dfrac{x_1}{1 + x_1} - a x_2\Big) \end{cases}$$

with parameters $a, b > 0$. In the first equation, the prey x_1 shows an expected exponential growth and a negative overpopulation factor $-x_1^2$. The last term is the loss due to predators, linear in x_2. If x_1 is low, $x_1 x_2/(1 + x_1) \approx x_2 x_1$, i.e. the deaths are proportional to the number of encounters between prey and predator. If the number of prey x_1 is high, $x_1 x_2/(1 + x_1) \approx x_2$, the number of deaths becomes independent of x_1 (there is a saturation as each predator is at its maximum eating capacity).

(a) Find the nullclines of the system.

(b) Two fixed points lie on the x_1 and x_2 axis. Find them and, by linearising the system, determine their stability properties, if possible.

(c) Prove that, for all values $a, b > 0$, there exists at least a fixed point with $x_1^*, x_2^* > 0$.

(d) A Hopf bifurcation can only occur when the trace of the linearized system changes sign. Use this fact to find the critical value $a_c(b)$ at which the Hopf bifurcation can occur at the fixed point (x_1^*, x_2^*). Is there a condition on b for the Hopf bifurcation to exist ?

(e) Check that the determinant of the linearised system is positive at that critical value. Together with the previous result, this implies that the eigenvalues are complex conjugate with zero real part supporting the hypothesis that a Hopf bifurcation is present.

(f) Visualise the bifurcation using Matlab. Use a fixed $b = 4$, then vary the value of a from $0.95 a_c(b)$ to $1.01 a_c(b)$. At each value of a, plot the nullclines, numerically find the fixed point, and plot two trajectories, one starting close to the fixed point, the other starting close to the origin. Is this Hopf bifurcation super or sub-critical ?
Help : To find the roots of a polynomial equation in Matlab use the `roots` command.

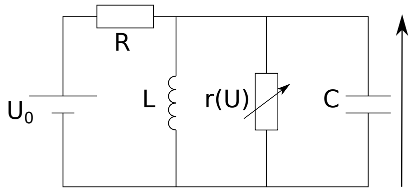
3 Van der Pol Oscillators

In the first half of the twentieth century, important research was done of the nonlinear dynamics of oscillators, motivated by the developpement of the radio and circuits with vacuum tubes. Many of these circuits have their dynamics governed by the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$. For this equation, the Liénard theorem states that if

- $f(x)$ and $g(x)$ are continuously differentiable $\forall x$
- $g(x)$ is an odd function
- $g(x) > 0$ for $x > 0$
- $f(x)$ is an even function
- $F(x) = \int_0^x f(u)\mathrm{d}u$ is odd and vanishes at $x = a > 0$
- For $0 < x < a$, it is $F(x) < 0$ while for $x > a$, $F(x) > 0$ and is non decreasing.
- $\lim\limits_{x \to \infty} F(x) = \infty$

then there is a unique stable limit cycle in the phase plane that surrounds the origin. One can understand this result qualitatively. In fact $-g(x)$ represents a nonlinear restoring force and $-f(x)\dot{x}$ a nonlinear damping force.

(a) A famous case of the Liénard equation is the Van der Pol equation. It can be derived from a circuit with a resistance, an inductance, a capacitor and a triode, which has a variable resistivity $r(U) \propto 1/U^2$, U being the applied voltage. The dynamics of the circuit is determined by the non-dimensionalised equation $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, with $\mu > 0$, i.e. the Van der Pol equation. Show that Liénard's theorem applies to this equation.



(b) Prove that the differential equation system

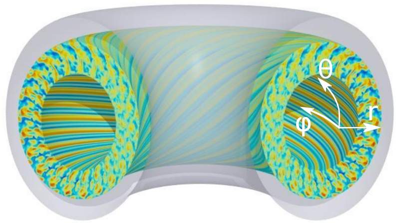
$$\begin{cases} \dot{x}_1 = x_2 - \mu(\frac{1}{3}x_1^3 - x_1) \\ \dot{x}_2 = -x_1 \end{cases}$$

is equivalent to the Van der Pol equation. Qualitatively draw the phase portrait.

(c) Suppose that $\mu \ll 1$. In this case the limit cycle, C, is approximately a circle centered at the origin. Use the divergence theorem $\oint_C \dot{\vec{x}} \cdot \vec{n}\, \mathrm{d}l = \iint_A \vec{\nabla} \cdot \dot{\vec{x}}\, \mathrm{d}A$ to get the radius.

4 L-H transition in tokamak plasmas

The mechanism underlying the transition from Low (L) to High (H) confinement mode in a tokamak plasma is still not yet fully understood. Generally, the improvement in confinement is associated to a locally reduced amplitude of turbulent fluctuations, which occurs when the power injected into the plasma exceeds a certain threshold that depends on the characteristics of the tokamak. The tokamak geometry is shown in the figure. The results of a numerical simulation of the turbulence in a tokamak is shown and the colors represent the amplitude of plasma density fluctuations. We call r, θ and φ the radial, poloidal and toroidal directions, respectively. Many different approaches have been attempted to describe the L-H transition.



The aim of this exercise is to study a model composed of two equations that describe plasma turbulence and its saturation, as proposed in *P. H. Diamond et al., Phys. Rev. Lett. 1994*. The first equation of the model can be written in the form :

$$\frac{1}{2}\frac{\partial \bar{E}}{\partial t} = \gamma_0 \bar{E} - a_1 \bar{E}^2 - a_2 \bar{E}\bar{U}$$

This equation represents the evolution of the energy associated with the turbulent fluctuations. Here $\bar{E} \equiv \left|\frac{\tilde{n}_k}{n_0}\right|^2$, where $\tilde{n}_k$ stands for the amplitude (standard deviation) of plasma density fluctuations, and n_0 the background density value. The terms $\gamma_0 \bar{E}$ and $-a_1 \bar{E}^2$ represent the linear growth of the instability driving the turbulence fluctuations and the turbulence saturation due to nonlinear stabilizing terms, respectively. The last term, $-a_2 \bar{E}\bar{U}$, describes the turbulence saturation due to the shearing of the turbulent eddies caused by the poloidal flow, being $\bar{U} \equiv \left|\frac{\partial \langle V_\theta \rangle}{\partial r}\right|^2$ the energy associated with poloidal flows, V_θ the poloidal velocity, and $\langle \cdot \rangle$ operator representing the average over θ.

(a) We now derive the second equation of the model to state the evolution of $\bar{U}$:

$$\frac{1}{2}\frac{\partial \bar{U}}{\partial t} = -\mu \bar{U} + a_3 \bar{E}\bar{U}$$

where $-\mu\bar{U}$ is the viscous damping of the poloidal flow and $a_3 \bar{E}\bar{U}$ is the drive of the poloidal flow due to turbulence (by means of the Reynold stress mechanism). Derive this equation starting from the conservation of poloidal momentum :

$$\frac{\mathrm{d}V_\theta}{\mathrm{d}t} = \frac{\partial V_\theta}{\partial t} + \left(V_r \frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}\right) V_\theta = -\frac{1}{r\rho_0}\frac{\partial p}{\partial \theta} - \mu V_\theta$$

where the pressure gradient and the viscous damping of the poloidal flow are taken into account (μ is constant). The quantity ρ_0 is the background plasma density.

Average the poloidal momentum conservation over the θ direction, keeping in mind that the domain is periodic in θ. Separate velocities in a part averaged on θ and in a fluctuating one :

$$V_\theta(t,r,\theta) = \langle V_\theta \rangle(t,r) + \tilde{V}_\theta(t,r,\theta) \quad , \quad V_r(t,r,\theta) = \langle V_r \rangle(t,r) + \tilde{V}_r(t,r,\theta) \approx \tilde{V}_r(t,r,\theta)$$

Furthermore, the first approximation you need to consider is :

$$\left|\left\langle \tilde{V}_\theta \frac{\partial}{\partial r}\tilde{V}_r \right\rangle\right| \ll \left|\frac{\partial}{\partial r}\left[\langle \tilde{V}_r \tilde{V}_\theta \rangle\right]\right|.$$

and use the assumption :

$$\frac{\partial^2}{\partial r^2}\left[\langle\tilde{V}_r\tilde{V}_\theta\rangle\right] = -a_3\bar{E}\frac{\partial\langle V_\theta\rangle}{\partial r},$$

The term $\langle\tilde{V}_r\tilde{V}_\theta\rangle$ is the "Reynold stress". Due to this term small-scale turbulence fluctuations can drive large-scale flows in the poloidal direction.

The set of coefficients in the model, $(\gamma_0, a_1, a_2, a_3, \mu) \geq 0$, depends on the type of instability in the plasma, and on the considered wavelength of the fluctuations.

We perform here an analysis of the system. Despite its simplicity, the model shows stationary solutions corresponding to the L-mode, where the fluctuation level is high and the poloidal flow is low, and to the H-mode, where the poloidal flow limits the amplitude of fluctuations.

(b) Choosing $E = a_1\bar{E}/\gamma_0$, $U = a_2\bar{U}/\gamma_0$ and $\tau = t\gamma_0$, reduce the system to two equations dependent only on the parameters $a = a_3/a_1$ and $b = \mu/\gamma_0$.

(c) Find the equilibrium points of the system. Distinguish the case $b > 0$ and $b = 0$. Discuss for each equilibrium point its physical meaning.

(d) Determine the nature (stability properties) of the equilibrium points. Consider b as a fixed parameter and vary a. Do you see a bifurcation ? Which bifurcation is it ? Discuss the physical meaning of this bifurcation and find the a value for which the L-H transition occurs.

(e) Trace a bifurcation diagram of E and U as a function of the control parameter a, considering $b = 1$.

(f) Integrate numerically the system, for different values of the control parameter a, above and below the bifurcation found in point (d). Consider $b = 1$. Verify that in the case of eigenvalues with an imaginary part, the convergence to the equilibrium point is oscillatory.

(g) Trace the phase space diagram in the particular case $a = 3$ and $b = 1$, and in the domain $0.1 < E < 2.1$ and $0.1 < U < 2.1$. Is there a globally stable equilibrium point ?

(h) Consider the case $a = 1$ and $b = 0$. In this case, the dynamics of the system includes no damping of the poloidal flows. Trace the phase space of the dynamical system in the domain $0.1 < E < 2.1$ and $0.1 < U < 2.1$. Comment on the behaviour of the system.