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Problem Set 6 : Two-Dimensional Models

1 Liapunov Function

Consider the system

$$\begin{cases} \dot{x}_1 = -\sin x_1 \cos x_2 \\ \dot{x}_2 = -\cos x_1 \sin x_2 \end{cases}$$

(a) Find the equilibrium points.

(b) Find a Liapunov function L for the equilibrium point $(0, 0)$.

(c) Plot the Liaponuv function. In Matlab, take advantage of the `meshgrid` and `surf` commands.

(d) Explain why the sign of $\vec{\nabla} L \cdot \vec{x}$ at a given point determines whether the trajectory is going towards or away from $(0, 0)$. Make a plot of $\vec{\nabla} L \cdot \vec{x}$ (in Matlab, use the command `contour`) and determine the biggest radius R such that any trajectory starting inside the disc centered at $(0, 0)$ of radius R remains inside the disc.

(e) For trajectories starting on discs centered at $(0, 0)$ of radius less than R, will the trajectories all converge towards $(0, 0)$?

2 Hamiltonian System

A particle moves under the effect of an inverse square force (such as gravity). The dynamics along the radial direction is determined by

$$\begin{cases} \dot{r} = \frac{p_r}{m} \\ \dot{p}_r = \frac{h^2}{r^3} - \frac{k}{r^2} \end{cases}$$

where m is the mass of the particle, h its angular momentum, and k the force constant.

(a) Find the Hamiltonian of the particle.

(b) For an attractive force ($k > 0$), identify and qualitatively plot the effective potential, i.e. the function $V_{\text{eff}} = V_{\text{eff}}(r)$ such that $\dot{p}_r = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}$. Find the local extrema and limits as $r \to 0$ and $r \to \infty$

(c) Draw the phase portrait of this system and describe the trajectories for the different values of the energy.

(d) Repeat questions (b) and (c) with $k < 0$.

3 Poincaré-Bendixson Theorem

Consider the system

$$\begin{cases} \dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + 5x_2^2) \\ \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{cases}$$

(a) Rewrite the differential equations by using the radial coordinates $\dot{r} = \dot{r}(r, \theta)$ and $\dot{\theta} = \dot{\theta}(r, \theta)$.

(b) Find a fixed point for the system and, by plotting the r and θ nullclines (e.g. with Matlab), show that it is unique.

(c) Determine the largest radius for which $\dot{r} \geq 0$ for all θ.

(d) Determine the smallest radius for which $\dot{r} \leq 0$ for all θ.

(e) Prove that there is a limit cycle and show its presence numerically.