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Problem Set 5 : Two-Dimensional Systems

1 Linearisation

Consider the two-dimensional system

$$\begin{cases} \dot{x}_1 = f_1 = \cos x_1 + \sin x_2 \\ \dot{x}_2 = f_2 = \cos x_1 \end{cases} \tag{1}$$

(a) Identify the x_2 nullclines.

(b) Identify the x_1 nullclines.

(c) Find all the fixed points.

(d) Linearise the system around the equilibrium points and, if possible, identify their nature (stable, unstable, center, stable spiral, unstable spiral . . .).

(e) Draw qualitatively the phase portrait by indicating the nullclines and arrow direction in each sector.

2 Linearisation II

Repeat the first exercise with the following system :

$$\begin{cases} \dot{x}_1 = x_1 x_2^2 - x_1^2 \\ \dot{x}_2 = x_2(x_2 - 1) \end{cases} \tag{2}$$

3 Epidemic Model

Some of the first epidemiological models were conceived by Kermarck and McKendrick in the 1920s. Nowadays a great range of more complicated models exist (we will explore them later in the semester). Among the models developed by Kermarck and McKendrick we consider the SIR model. The letter S stands for susceptible (healthy individuals that can get infected), I for infected and R for individuals that have recovered from the illness (if we suppose that this is not a deadly disease). The model is expressed as

$$\begin{cases} \dot{S} = -\alpha S I \\ \dot{I} = \alpha S I - \beta I \\ \dot{R} = \beta I \end{cases}$$

The susceptible individuals get infected at a rate proportional to the encounters they have with infected individuals explaining the term $\alpha S I$, with $\alpha > 0$. Also the rate at which people recover from the disease is proportional to the number of infected individuals, explaining βI, with $\beta > 0$. In this model people who recover from the disease become immune and cannot get infected again. In cases like smallpox this is a good model. Here it is assumed that the total population stays constant, i.e. births, deaths, and migrations occur on a time scale much slower than the spread of the disease.

(a) Show that indeed the total population is constant. Then, show that the dynamics of the R variable is determined by the two other variables, meaning that we can eliminate R, and reduce the SIR model to a two-dimensional model.

(b) Find the fixed points of the system. Identify, when possible, the stable and unstable manifolds of the linearised system.

(c) Plot qualitatively the phase portrait by indicating the nullclines and the arrow direction in each sector. Find the physical interpretation of this sketch. Complete the analysis with a numerical plot.

(d) An epidemic occurs when the number of infected people increases initially. Find the initial conditions for an epidemic to occur. Discuss the physical meaning of varying the parameters α and β.

(e) Find analytically the trajectories $I = I(S)$ and compare them with the numerical solutions.

(f) Find and discuss the limit (S_f, I_f) of the system as $t \to +\infty$.