18 March 2025

Problem Set 4 : Bifurcations and Two-Dimensional Systems

1 Numerical Phase Portrait

In this problem, we numerically produce the phase portrait of a two-dimensional system of differential equations using Matlab. Consider the differential equation

$$\begin{cases} \dot{x}_1 = \cos x_1 + \sin x_2 \\ \dot{x}_2 = \cos x_1 \end{cases}$$

(a) Use Matlab's `meshgrid` and `quiver` functions to plot the vector field associated with the differential equation. Take the periodicity into consideration to determine the area on which to plot.

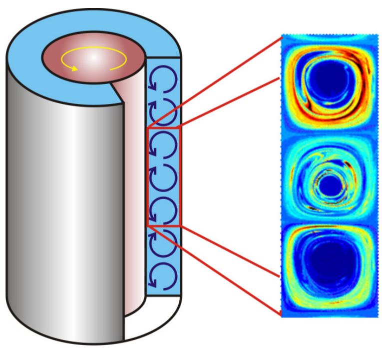
(b) Sometimes, due to strong variations in the vector field, some arrows are very small and hard to see. In some cases we are more interested in the direction of the vector rather than in the norm. Plot the vector field again, having renormalized the vectors to the unitary length.

(c) Another way to visualise a vector field is to plot the streamlines. Generate a grid of coordinates, which we define as the initial positions of the streamlines. For each of these initial conditions integrate the differential equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ up to a certain time t_f. Plot each trajectory. You can adjust t_f and refine or coarse the initial condition grid, add different symbols to the beginning and end of the trajectories to indicate the directions, alternate colors...

(d) Based on the numerical phase portrait you obtained, discuss qualitatively the behaviour of the solution (fixed points, their nature...).

2 Bifurcation in a Fluid

The Taylor-Couette flow occurs when a fluid is placed in the gap between two co-axial cylinders rotating around their axis at different speeds. If the external cylinder is kept fixed and the inner one is rotating sufficiently slowly, a stable stationary flow is observed. Analytical calculations show that the azimuthal velocity of the flow is $v_\theta = C_1 r + C_2/r$. When the angular velocity of the inner cylinder exceeds a threshold, a smaller secondary flow appears on top of the fluid stationary flow. Its amplitude, $A(t)$, follows the simplified equation $\dot{A} = h + \epsilon A - gA^3 - kA^5$. The sign of A indicates the direction of the perturbed flow. The symmetry breaking parameter h represents possible imperfections of the system. The parameter k is always positive, preventing A from diverging. The system is said supercritical if $g > 0$ and subcritical if $g < 0$.



(a) First assume $g = 0$ and $h = 0$. Normalise the system so that it depends only on one parameter. Then plot qualitatively the bifurcation diagram for that parameter. Identify the type of bifurcation.

(b) To see the effect of an imperfection on our system, take $h > 0$ (still with $g = 0$). Find the critical value $\epsilon = \epsilon_c(h, k)$ at which the number of fixed points changes. Plot qualitatively the bifurcation diagram A^* in terms of ϵ. Discuss the limit of h going to zero.

(c) Neglect again the imperfections ($h = 0$). Using $\epsilon > 0$, find the fixed points. Then, discuss qualitatively their stability properties and show that the sign of g cannot change them.

BONUS

(d) Still with $h = 0$, set $\epsilon < 0$ and make a qualitative plot of the bifurcation diagram when g is varied. In in the (k, g) plane, delimit the regions that have a different number of stationary points, finding the analytical equation that identifies these regions.