4 March 2025

Problem Set 3 : Bifurcations and Reminder of Linear Algebra

1 Laser Threshold

We model the laser dynamics with the equation $\dot{n} = GnN - kn$, where $n(t) > 0$ is the number of photons in the laser field. In the laser medium the photons decay exponentially due to mirror transmission, scattering, and so on, which explains the $-kn$ term. The photons are created by stimulated emission, when a photon interacts with an excited atom, which de-excites and emits a photon with the same phase, explaining the term $+GnN$. ($N(t) > 0$ is the number of excited atoms, which depends on time, and $G > 0$ is the gain coefficient.) We initially assume that $N = N_0 - \alpha n$, where $N_0 > 0$ is the number of atoms that the pump keeps excited in the absence of photons. The parameter $\alpha > 0$ is a proportionality constant that indicates the decrease of the number of excited atoms because of the presence of photons.

(a) Find the differential equation for n and rescale n so that the laser dynamics depend only on one parameter.

(b) Find the equilibrium points of n. What are their stability properties ? Identity the bifurcation type as you vary the identified parameter and draw the bifurcation diagram. When are the stationary points physically acceptable ?

(c) What is the threshold condition to emit laser light ?

To improve the model, the number of excited atoms is assumed to follow $\dot{N} = -GnN - fN + p$. The $-GnN$ is the counterpart to the stimulated emission. $f > 0$ is the decay rate for spontaneous emission. p, positive or negative, is the pump strength. We use the quasi-static approximation $\dot{N} \approx 0$, assuming that N relaxes much faster than n.

(d) Derive the new differential equation for n and renormalise the time and the density so that the model depends only on one parameter.

(e) Find the equilibrium points as well as their stability properties. What is the critical pump strength threshold ?

2 Laser threshold with an advanced model

For a laser model considerably improved with respect to the one considered in the first exercise, we introduce the electric field E, the mean polarisation P and the population inversion D. They follow the Maxwell-Bolch equations :

$$\begin{cases} \dot{E} = \kappa(P - E) \\ \dot{P} = \gamma_1(ED - P) \\ \dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP) \end{cases}$$

The constant $\kappa > 0$ is the decay rate in the laser cavity due to beam transmission. $\gamma_1, \gamma_2 > 0$ are the decay rates of the atomic polarization and population inversion. λ, positive or negative, is the pumping strength. If $\gamma_1, \gamma_2 \gg \kappa$ then the adiabatic elimination can be used, where $\dot{P} = 0$ and $\dot{D} = 0$. Using the adiabatic approximation, simplify the evolution equation of E. Then, plot the bifurcation diagram of E with respect to the parameter λ.

3 Exponential of a Matrix and Systems of Linear Differential Equations

As a warm up for the analysis of multi-dimensional non-linear systems, we recall how multi-dimensional linear systems can be solved and how the exponential of a matrix can be evaluated.

(a) A square matrix N is nilpotent if there exists $m_* \in \mathbb{N}$ such that $N^m = 0,\ \forall m > m_*$. Compute the powers of the upper shift matrix of size 4, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Use a computer if necessary. Notice the pattern ? Is this a nilpotent matrix ?

(b) Find the exponential of the upper shift matrix N.
Reminder : For matrices, $\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!} = I + M + \frac{1}{2}M^2 + \dots$

(c) What is the exponential of a diagonal matrix ?

(d) Compute the exponential of a Jordan block J_λ, that is

$$\exp(J_\lambda) = \exp\left(\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}\right)$$

Hint : If two matrices, A and B, commute (i.e. $[A, B] = AB - BA = 0$), then $\exp(A + B) = \exp(A)\exp(B)$.

(e) Show that the matrix

$$M = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

can be written as a Jordan block under a basis transformation and evaluate the proper transition matrix.
Reminder : Suppose a 3 by 3 matrix M can be written as J_λ by changing basis, i.e. there exists a transition matrix P such that $M = PJ_\lambda P^{-1}$. P is the 3×3 matrix whose columns are the three new basis vectors side by side. To find the basis vectors notice that if M, represented in the $\{v_1, v_2, v_3\}$ basis, is written

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

then, by definition,

$$\begin{cases} Mv_1 = \lambda v_1 \\ Mv_2 = \lambda v_2 + v_1 \\ Mv_3 = \lambda v_3 + v_2 \end{cases}$$

Indeed, the basis vectors can be found by solving

$$\begin{cases} (M - \lambda I)v_1 = 0 \\ (M - \lambda I)v_2 = v_1 \\ (M - \lambda I)v_3 = v_2 \end{cases}$$

(f) Using the properties of the matrix exponential, explicitly compute the solution of the system $\dot{\vec{x}} = M\vec{x}$, with $\vec{x}(t_0) = \vec{x}_0 = [x_{01}, x_{02}, x_{03}]^T$.
Reminder : $\exp(A\ B\ A^{-1}) = A\ \exp(B)\ A^{-1}$