Quantum trajectories

I. Stochastic Schrödinger equation

1) Introduction

duration Δt $\quad \{\hat{M}_\mu\} \to$ measurement operators

update rule: $|\psi_c(t+\Delta t)\rangle = \dfrac{\hat{M}_\mu |\psi_c(t)\rangle}{\sqrt{p_\mu}}$, conditioned on outcome μ.

- $\hat{M}_\mu = \sqrt{\Delta t}\, \hat{L}_\mu \quad \mu \geq 1$
- $\hat{M}_0 = \hat{\mathbb{1}} - i\hat{H}\Delta t - \sum_{\mu\geq 1} \dfrac{\hat{L}_\mu^\dagger \hat{L}_\mu}{2}\Delta t = \hat{\mathbb{1}} - i\hat{H}_{eff}\Delta t$

$$p_\mu = \langle\psi_c(t)|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi_c(t)\rangle = \begin{cases} \langle\psi_c(t)|\hat{L}_\mu^\dagger \hat{L}_\mu|\psi_c(t)\rangle\cdot\Delta t & \text{for } \mu\geq 1 \\ 1 - \sum_{\mu\geq 1}\langle\psi_c(t)|\hat{L}_\mu^\dagger\hat{L}_\mu|\psi_c(t)\rangle & \mu = 0\end{cases}$$

2) Unraveling

From the point of view of the "environment": $N_\mu(t)$ number of "clicks" of detector μ, in the interval $[0,t]$

$N_\mu(t)$: classical, random variable.

Let $\Delta t \to dt$: $\quad dN_\mu(t) = 1$ if detector μ clicked between t and $t+dt$

$= 0$ otherwise

$$dN_0(t) = 1 - \sum_{\mu\geq 1} dN_\mu(t)$$

dN_μ is a classical random process: $\begin{cases} dN_\mu = 1 & \text{with probability } p_\mu = \langle L_\mu^\dagger L_\mu\rangle\cdot dt \\ dN_\mu = 0 & \dots\ 1 - p_\mu\end{cases}$

ensemble average $\to$ $\overline{dN_\mu} = p_\mu$

"point process": $\quad dN_\mu dN_\nu = \delta_{\mu\nu} dN_\mu \quad , \quad dN_\mu dt \ll dt$

3) Derivation

$$|\psi_c(t+dt)\rangle = \sum_\mu dN_\mu(t)\frac{\hat{M}_\mu|\psi_c(t)\rangle}{\sqrt{p_\mu}}$$

Separate $\mu=0$ and $\mu\geq 1$

$$|\psi_c(t+dt)\rangle = \left(1-\sum_{\mu\geq 1} dN_\mu(t)\right)\frac{\hat{M}_0|\psi_c\rangle}{\sqrt{1-\underbrace{\sum_{\mu\geq 1} p_\mu}_{O(dt)}}} + \underbrace{\sum_{\mu\geq 1} dN_\mu(t)\frac{\hat{M}_\mu|\psi_c(t)\rangle}{\sqrt{p_\mu}}}_{\sum_{\mu\geq 1} dN_\mu(t)\frac{\hat{L}_\mu|\psi_c(t)\rangle}{\sqrt{\langle L_\mu^+ L_\mu\rangle}}}$$

$$|\psi_c(t+dt)\rangle = \left(1-\sum_{\mu\geq 1} dN_\mu(t)\right)\cdot\left(1+\frac{dt}{2}\sum_\mu\langle\hat{L}_\mu^+\hat{L}_\mu\rangle\right)\left(\hat{1}-i\hat{H}dt-\frac{dt}{2}\sum_\mu\hat{L}_\mu^+\hat{L}_\mu\right)|\psi_c(t)\rangle$$
$$+\sum_{\mu\geq 1} dN_\mu(t)\frac{\hat{L}_\mu|\psi_c(t)\rangle}{\sqrt{\langle\hat{L}_\mu^+\hat{L}_\mu\rangle}}$$

$$|\psi_c(t+dt)\rangle - |\psi_c(t)\rangle = -i\hat{H}dt|\psi_c\rangle - \frac{dt}{2}\sum_\mu\left(\hat{L}_\mu^+\hat{L}_\mu - \langle\hat{L}_\mu^+\hat{L}_\mu\rangle\right)|\psi_c\rangle$$
$$+\sum_\mu dN_\mu(t)\left(\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^+\hat{L}_\mu\rangle}}-\hat{1}\right)|\psi_c\rangle$$

$$\boxed{d|\psi_c\rangle = \left(-i\hat{H}-\frac{1}{2}\sum_\mu\hat{L}_\mu^+\hat{L}_\mu - \langle\psi_c|\hat{L}_\mu^+\hat{L}_\mu|\psi_c\rangle\right)dt\,|\psi_c(t)\rangle + \sum_\mu dN_\mu\left(\frac{\hat{L}_\mu}{\sqrt{\langle\hat{L}_\mu^+\hat{L}_\mu\rangle}}-1\right)|\psi_c\rangle}$$

SSE

Remarks:
- Solution to SSE $\{N_\mu(t), |\psi_c(t)\rangle\}$ quantum trajectory
- Non-linear

II. Interpretation

1) Lindblad equation

Consider $\overline{|\psi_c(t)\rangle\langle\psi_c(t)|}$ ← average over the random process dN_μ

$$\overline{|\psi_c(t+dt)\rangle\langle\psi_c(t+dt)|} = \overline{|\psi_c(t)\rangle\langle\psi_c(t)|} + \overline{d|\psi_c\rangle\langle\psi_c(t)|} + \overline{|\psi_c(t)\rangle\, d\langle\psi_c|}$$

$$+ \overline{d|\psi_c\rangle\, d\langle\psi_c|}$$

not of order 2

$$d\hat\rho = \left[\left(-i\hat H_{eff} + \frac{1}{2}\sum_\mu \langle \hat L_\mu^+ \hat L_\mu\rangle\right)dt\, |\psi_c(t)\rangle\langle\psi_c(t)|\right] + \sum_\mu \overline{dN_\mu}\left(\frac{\hat L_\mu |\psi_c\rangle\langle\psi_c|}{\sqrt{\langle L_\mu^+ L_\mu\rangle}} - |\psi_c\rangle\langle\psi_c|\right) +$$

$$\overline{dN_\mu} = \langle \hat L_\mu^+ \hat L_\mu\rangle dt$$

$$|\psi_c(t)\rangle\langle\psi_c(t)|\left(+i\hat H_{eff}^+ + \frac{1}{2}\sum_\mu \langle \hat L_\mu^+ \hat L_\mu\rangle\right)dt + \sum_\mu \overline{dN_\mu}\left(\frac{|\psi_c\rangle\langle\psi_c|\hat L_\mu^+}{\sqrt{\langle \hat L_\mu^+ \hat L_\mu\rangle}} - |\psi_c\rangle\langle\psi_c|\right) +$$

$$\overline{dN_\mu} = \langle \hat L_\mu^+ \hat L_\mu\rangle dt$$

$$\rightarrow \sum_{\mu\mu'} \overline{dN_\mu dN_{\mu'}}\left(\frac{\hat L_\mu}{\sqrt{\langle \hat L_\mu^+ \hat L_\mu\rangle}} - \mathbb{1}\right)|\psi_c\rangle\langle\psi_c|\left(\frac{\hat L_{\mu'}^+}{\sqrt{\langle \hat L_{\mu'}^+ \hat L_{\mu'}\rangle}} - \mathbb{1}\right)$$

- $\overline{dN_\mu dN_{\mu'}} = \delta_{\mu\mu'}\overline{dN_\mu}$
- $\overline{dN_\mu} = \langle \hat L_\mu^+ \hat L_\mu\rangle dt$

last term: $\displaystyle\sum_\mu dt\langle \hat L_\mu^+ L_\mu\rangle\left(\frac{-\hat L_\mu|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi| L_\mu^+}{\sqrt{\langle \hat L_\mu^+ \hat L_\mu\rangle}} + \frac{\hat L_\mu |\psi\rangle\langle\psi|\hat L_\mu^+}{\sqrt{\hat L_\mu^+ \hat L_\mu}} + |\psi\rangle\langle\psi|\right)$

$$\boxed{d\hat\rho = -i\left[\hat H_{eff}\hat\rho - \hat\rho \hat H_{eff}^+\right] + \sum_\mu \hat L_\mu \hat\rho \hat L_\mu^+}$$

Lindblad equation

2) Monte-Carlo wavefunction algorithm

$\dim \mathcal{H} = N$ $\qquad \hat\rho \sim O(N^2)$ entries

propagating $\hat\rho \sim O(N^4)$

pure state $|\psi\rangle \sim O(N)$ entries

propagation of $|\psi\rangle \sim O(N^2)$

M different trajectories → recovering $\hat\rho = \overline{|\psi_c\rangle\langle\psi_c|} \sim O(MN^2)$

$M \ll N^2$ ⊕ parallel

- Naive algorithm : . define δt time step
  . each δt : draw random number $R \in [0,1]$

  jump probability : $p_\mu = \langle \psi_c | \hat{L}_\mu^\dagger \hat{L}_\mu | \psi_c \rangle \delta t$

  total jump proba. : $p = \sum_\mu p_\mu$

  . <u>if $R < p$:</u> jump happens , determine randomly which one $\{\mu\}_{\mu \geq 1}$

  Apply corresponding $\hat{L}_\mu |\psi_c(t)\rangle$ + normalize to obtain $|\psi_c(t+\delta t)\rangle$

  Record the jump event on "channel" .

  . <u>if $R > p$:</u> determine $\delta|\psi_c\rangle = \left[-i\hat{H} - \frac{1}{2} \sum_\mu \left(\hat{L}_\mu^\dagger \hat{L}_\mu - \langle \hat{L}_\mu^\dagger \hat{L}_\mu \rangle \right) \right] \delta t |\psi_c\rangle$

  $$|\psi_c(t+\delta t)\rangle = |\psi_c(t)\rangle + \delta|\psi_c\rangle$$

- Clever algorithm : use non-normalized wavefunction for the no-jump evolution :

  $$\frac{d|\tilde{\psi}\rangle}{dt} = -\left(i\hat{H} + \frac{1}{2} \underbrace{\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu}_{\to \text{ decrease of } \langle \tilde{\psi} | \tilde{\psi} \rangle} \right) |\tilde{\psi}\rangle$$

  draw random R , evolve until $\langle \tilde{\psi} | \tilde{\psi} \rangle = R$

  ↳ then jump (like previously)

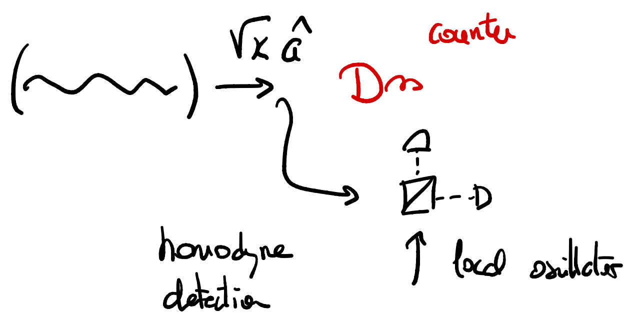
3) Different types of unravelings

Reminder : unitary transformations on Kraus operators : $\hat{K}_\nu = \sum_{\mu\nu} u_{\mu\nu} \hat{M}_\mu$

$$\Rightarrow \sum_\nu \hat{K}_\nu \hat{\rho} \hat{K}_\nu^\dagger = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger \; ; \; \sum_\nu \hat{K}_\nu^\dagger \hat{K}_\nu = \hat{1} = \sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu$$

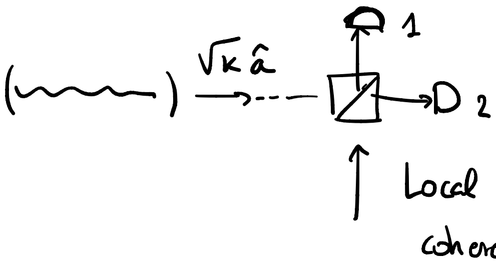
Lindblad equation : similar

example : photon detection ,



III - Weak Continuous measurements

1) Homodyning



$$\hat L = \sqrt{\kappa}\,\hat a \longrightarrow \sqrt{\kappa}\,\frac{\hat a - \alpha \hat 1}{\sqrt 2} = \hat c_1$$

$$\sqrt{\kappa}\, i\, \frac{\hat a + \alpha \hat 1}{\sqrt 2} = \hat c_2$$

Transformations

$$\hat M_0^\dagger \hat M_0 + \hat M_1^\dagger \hat M_1 = \hat 1$$

$$(\hat 1 - i\hat H_{eff}\delta t)^\dagger (\hat 1 - i\hat H_{eff}\delta t) + \kappa \hat a^\dagger \hat a\, \delta t = \hat 1$$

$$i(\hat H_{eff}^\dagger - \hat H_{eff}) + \kappa \hat a^\dagger \hat a = 0$$

$$\hat H_{eff} = \hat H - i\frac{\kappa \hat a^\dagger \hat a}{2}$$

now:

$$\hat M_0^\dagger \hat M_0 + \hat M_1^\dagger \hat M_1 + \hat M_2^\dagger \hat M_1 = \hat 1$$

$$(\hat 1 - i\hat H_{eff}\delta t)^\dagger (\hat 1 - i\hat H_{eff}\delta t) + \frac{\kappa}{2}\delta t\left(\hat a^\dagger \hat a + |\alpha|^2 - \cancel{\hat a\alpha^*} - \cancel{\hat a^\dagger \alpha}\right)$$

$$+ \frac{\kappa}{2}\delta t\left(\hat a^\dagger \hat a + |\alpha|^2 + \cancel{\hat a\alpha^*} + \cancel{\hat a^\dagger \alpha}\right)$$

$$\hat H_{eff} = \hat H - \frac{i\kappa}{2}\left(\hat a^\dagger \hat a - |\alpha|^2\right)$$

constant is irrelevant (normalized out)

For detector 1:

$$p_1 = \delta t \langle \hat c_1^\dagger \hat c_1 \rangle = \frac{\kappa \delta t}{2}\left(|\alpha|^2 + \langle \hat a^\dagger \hat a\rangle - \langle \alpha \hat a^\dagger + \alpha^* \hat a\rangle\right)$$

let $\alpha = |\alpha| e^{i\phi}$ and $\hat x_\phi = e^{i\phi}\hat a^\dagger + \hat a e^{-i\phi}$

$$p_1 = \frac{\kappa\delta t}{2}\left(|\alpha|^2 + \langle \hat a^\dagger \hat a\rangle - |\alpha| \langle \hat x_\phi\rangle\right)$$

$$p_2 = \frac{\kappa\delta t}{2}\left(|\alpha|^2 + \langle \hat a^\dagger \hat a\rangle + |\alpha| \langle \hat x_\phi\rangle\right)$$

let $|\alpha|$ is very large: $|\alpha|^2 \gg \langle \hat a^\dagger \hat a\rangle$

Integrate $\int_0^{\delta T} dN_1$: chose δT such that $\langle \hat x_\phi\rangle(t+\varepsilon) \sim \langle \hat x_\phi\rangle(t)$ for $\varepsilon \leqslant \delta T$

$$|\psi_c(t+\varepsilon)\rangle \sim |\psi_c(t)\rangle$$
$$\varepsilon \lesssim \delta\tau$$

- Let $I_{1,2} = \dfrac{\delta N_1}{\delta \tau}$: photocurrent

- Large number of counts, uncorrelated (Poisson) distributed $\rightarrow$ Gaussian

$$\delta N_1 = \overline{\delta N_1} + \delta W_1$$

$$\overline{\delta N_1} = \int_0^{\delta\tau} dN_1 = \frac{\varkappa}{2}\delta\tau \left(|\alpha|^2 + \langle \hat{X}_\phi\rangle(0) \cdot |\alpha| \right)$$

$$\overline{\delta W_1} = 0 \quad \text{and} \quad \overline{\delta W_1^2} = \overline{\delta N_1} \sim \frac{\varkappa |\alpha|^2 \delta\tau}{2}$$

So: $$dN_1(t) = \frac{\varkappa|\alpha|^2}{2}\left(1 + \frac{\langle \hat{X}_\phi\rangle(t)}{|\alpha|}\right) dt + |\alpha| \sqrt{\frac{\kappa}{2}}\, dW$$

dW : Wiener process

Langevin

$$I_{1,2}(t) = \frac{\kappa|\alpha|^2}{2}\left(1 \mp \frac{\langle \hat{X}_\phi\rangle(t)}{|\alpha|}\right) + |\alpha|\sqrt{\frac{\kappa}{2}}\, \xi_{1,2}(t)$$

$$\xi(t)\,dt = dW$$

<u>Balanced homodyne detection</u> record $I_2 - I_1$

$$I_{bh}(t) = \varkappa |\alpha| \langle \hat{x}_\phi\rangle(t) + |\alpha|\sqrt{\kappa}\, \xi(t)$$

$$\xi(t) = \frac{\xi_2 - \xi_1}{\sqrt{2}}$$

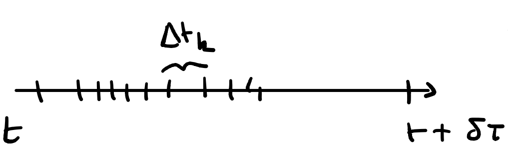
<u>Remarks</u>
- unbalanced also works,
- one can also use heterodyne detection:

$$\text{LO} \rightarrow \alpha e^{i\Omega t}$$

2) Quantum state diffusion

During $\delta\tau$

$$|\psi_c(t+\delta\tau)\rangle = \prod_i \hat{M}_{\mu_i}(\Delta t_i)\, |\psi_c(t)\rangle$$



μ_i

$$\rightsquigarrow \quad |\tilde{\psi}_c(t+\delta T)\rangle \sim \hat{M}_0(\delta T)\cdot \hat{c}_1^{m_1}\hat{c}_2^{m_2}|\tilde{\psi}_c(t)\rangle$$

conditioned on m_1 clicks on 1

m_2 clicks on 2

$$\hat{c}_1^{m_1} = \left(\sqrt{\frac{\kappa}{2}}\right)^{m_1}(\hat{a}-\alpha)^{m_1} = \left(\sqrt{\frac{\kappa}{2}}(-\alpha)\right)^{m_1}\cdot\left(\hat{\mathbb{1}}-\frac{\hat{a}}{\alpha}\right)^{m_1}$$

- $\left[\sqrt{\kappa/2}\,|\alpha|\right]^{m_1}$ → normalized act
- $(-e^{i\phi})^{m_1}$ → the only object relevant $|\psi_c\rangle\langle\psi_c|$

$$|\tilde{\psi}_c(t+\delta T)\rangle = \left(\hat{\mathbb{1}} - i\hat{H}_{eff}\delta\tau\right)\left(\hat{\mathbb{1}}-\frac{m_1\hat{a}}{\alpha}\right)\left(\hat{\mathbb{1}}+\frac{m_2\hat{a}}{\alpha}\right)|\tilde{\psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{1}} - i\hat{H}_{eff}\delta\tau + \frac{\hat{a}}{\alpha}(m_2-m_1)\right)|\tilde{\psi}_c(t)\rangle$$

$$= \left(\hat{\mathbb{1}} - i\hat{H}_{eff}\delta t + \frac{\hat{a}}{\alpha}\left[\kappa\frac{|\alpha|^2}{2}\left(1+\frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau + |\alpha|\sqrt{\frac{\kappa}{2}}\delta W_2 - \frac{\kappa|\alpha|^2}{2}\left(1-\frac{\langle x_\phi\rangle}{|\alpha|}\right)\delta\tau - |\alpha|\sqrt{\frac{\kappa}{2}}\delta W_1\right]\right)|\tilde{\psi}_c\rangle$$

$$\boxed{d|\tilde{\psi}_c\rangle = \left(-i\hat{H}_{eff}dt + \hat{a}e^{-i\phi}\left(\kappa\langle\hat{x}_\phi(t)\rangle dt + \sqrt{\kappa}\,dW\right)\right)|\tilde{\psi}_c\rangle}$$

also: $$d|\tilde{\psi}_c\rangle = \left(-i\hat{H}_{eff} + \frac{I_{hh}}{\alpha}\hat{a}e^{-i\phi}\right)dt\,|\tilde{\psi}_c\rangle$$

Normalization:

$$\langle\tilde{\psi}_c(t+dt)|\tilde{\psi}_c(t+dt)\rangle = 1 + \langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle + \langle d\psi|d\psi\rangle$$

$\vdots$

$$d|\psi_c\rangle = \left[-i\hat{H}_{eff}dt + \kappa\langle\hat{x}_\phi(t)\rangle dt\left(\hat{a}e^{-i\phi} - \frac{\langle\hat{x}_\phi\rangle(t)}{2}\right) + \kappa\,dW\left(ae^{-i\phi} - \frac{\langle\hat{x}_\phi\rangle}{2}\right)\right]|\psi_c\rangle$$

Remarks:

- read as drift - diffusion equation in the Hilbert space
- fully general (equivalent to dich - based SSE)