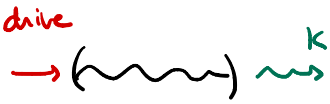
Damped harmonic oscillator

I. Introduction

Single mode of the e.m field 

- Hamiltonian: $\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \sqrt{\kappa}\,\varepsilon\left(\hat{a}\, e^{i\omega_e t} + h.c.\right)$

  in a frame rotating at ω_e : $\hat{U} = e^{i\omega_e t\cdot \hat{a}^\dagger \hat{a}}$

  $$\rightarrow \hat{H} = \Delta \hat{a}^\dagger \hat{a} + \sqrt{\kappa}\,\varepsilon\,(\hat{a} + \hat{a}^\dagger) \qquad \Delta = \omega_c - \omega_e$$

- jump : $\hat{L} = \sqrt{\kappa}\,\hat{a}$

  Lindblad equation : $\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] + \kappa\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}\right)$

II - Field Amplitude

$$\langle \hat{a} \rangle = \mathrm{Tr}(\hat{\rho}\hat{a})$$

$$\frac{d}{dt}\langle \hat{a}\rangle = \frac{d}{dt}\mathrm{Tr}[\hat{\rho}\hat{a}] = \mathrm{Tr}(\dot{\hat{\rho}}\hat{a})$$

$\left\{ \mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) \right.$

$$\mathrm{Tr}\left[\hat{a}\left(-i[\hat{H},\hat{\rho}] + \kappa\left(\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{\rho}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}^\dagger\hat{a}\hat{\rho}\right)\right)\right]$$

- $-i\,\mathrm{Tr}\left(\hat{a}[\hat{H},\hat{\rho}]\right) = -i\,\mathrm{Tr}\left(\hat{a}\hat{H}\hat{\rho} - \hat{a}\hat{\rho}\hat{H}\right) = -i\,\mathrm{Tr}\left((\hat{a}\hat{H} - \hat{H}\hat{a})\hat{\rho}\right)$

  $= -i\langle[\hat{a},\hat{H}]\rangle$

  $-i\langle[\hat{a},\ \Delta\hat{a}^\dagger\hat{a} + \sqrt{\kappa}\,\varepsilon(\hat{a} + \hat{a}^\dagger)]\rangle$

  $= -i\Delta\langle\hat{a}\rangle - i\sqrt{\kappa}\,\varepsilon$

- $\kappa\,\mathrm{Tr}\left(\hat{a}\hat{a}\hat{\rho}\hat{a}^\dagger - \frac{1}{2}\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \frac{1}{2}\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right)$

  $= \kappa\,\mathrm{Tr}\left(\cancel{\hat{a}^\dagger\hat{a}\hat{a}\hat{\rho}}\right) - \frac{\kappa}{2}\mathrm{Tr}\left(\cancel{\hat{a}^\dagger\hat{a}\hat{a}\hat{\rho}}\right) - \frac{\kappa}{2}\mathrm{Tr}\left(\hat{a}\hat{a}^\dagger\hat{a}\hat{\rho}\right)$

  $\hat{a}\hat{a}^\dagger = (\cancel{\hat{a}^\dagger\hat{a}} + 1)$

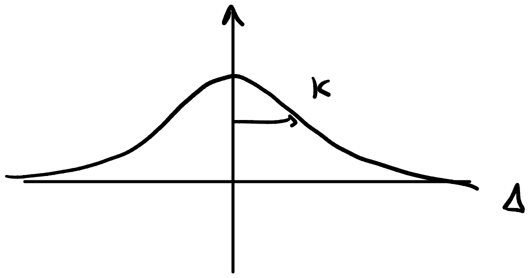
  $-\frac{\kappa}{2}\langle\hat{a}\rangle$

$$\boxed{\langle\dot{\hat{a}}\rangle = -i\Delta\langle\hat{a}\rangle - \frac{\kappa}{2}\langle\hat{a}\rangle - i\sqrt{\kappa}\,\varepsilon}$$

Steady state: $\langle \dot{\hat{a}} \rangle = 0 \Rightarrow \langle \hat{a} \rangle = \dfrac{-i\sqrt{\kappa}\,\varepsilon}{i\Delta + \kappa/2}$

$$\langle a \rangle = \frac{\sqrt{\kappa}\,\varepsilon}{-\Delta + i\kappa/2}$$

intensity: $$|\langle a \rangle|^2 = \frac{\kappa\,|\varepsilon|^2}{\Delta^2 + \kappa^2/4}$$



$|\langle a \rangle|^2 \sim$ # of photons

$= \dfrac{2|\varepsilon|^2}{\kappa}$ $\qquad |\varepsilon|^2$: photon flux $\Leftrightarrow$ laser intensity

$\Rightarrow \varepsilon$: laser amplitude

Remark: $\hat{H} \longrightarrow \hat{H}_{eff} = \left(\Delta - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}$ not Hermitian

III. Coherent state decay.

consider $|\alpha\rangle$: $\hat{L}|\alpha\rangle = \sqrt{\kappa}\,\alpha|\alpha\rangle$

Extended Hilbert space: $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$

photon in the environment

$$\hat{U}\left(|\alpha\rangle \otimes |0\rangle\right) = \underbrace{\left(\hat{M}_0|\alpha\rangle\right)}_{\text{no jump}} \otimes |0\rangle + \left(\hat{M}_1|\alpha\rangle\right) \underbrace{\otimes |1\rangle}_{\text{jump}}$$

$$= \left[\left(\hat{1} - i\Delta t\,\hat{H} - \Delta t\,\hat{J}\right)|\alpha\rangle\right] \otimes |0\rangle + \left(\hat{M}_1|\alpha\rangle\right) \otimes |1\rangle$$

$$= \left[\left(\hat{1} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}\right)|\alpha\rangle\right] \otimes |0\rangle + \left(\sqrt{\Delta t\,\kappa}\,\alpha|\alpha\rangle\right) \otimes |1\rangle$$

Measure the environment:

- 1 is obtained: $|\alpha\rangle \longrightarrow |\alpha\rangle$
- 0 is obtained: $\underbrace{\left(\hat{1} - i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}\right)}_{e^{-i\Delta t\left(\omega_c - \frac{i\kappa}{2}\right)\hat{a}^\dagger\hat{a}}}|\alpha\rangle$

if $\kappa = 0$: $e^{-i\Delta t\,\omega_c \hat{a}^\dagger \hat{a}} |\alpha\rangle$ free evolution

$$= |\alpha e^{-i\Delta t \cdot \omega_c}\rangle$$

with κ finite: $e^{-\Delta t \kappa/2\, \hat{a}^\dagger \hat{a}} |\alpha(t)\rangle$

$$= \mathcal{N} \cdot |e^{-\Delta t \kappa/2} \alpha(t)\rangle$$

IV- Alternative descriptions

1) Photon number

$|n\rangle$ Fock state $\quad \mathcal{H}_S = \mathrm{Sp}\{|n\rangle, n \in \mathbb{N}\}$

$$\langle n|\hat{\rho}|m\rangle = \rho_{nm}$$

No external drive: $\hat{H} = \omega_c \hat{a}^\dagger \hat{a}$

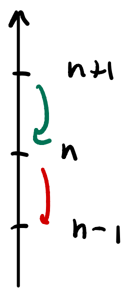
$$\dot{\rho}_{nm} = \frac{d}{dt}\langle n|\hat{\rho}|m\rangle = \langle n|\dot{\hat{\rho}}|m\rangle$$

$$= -i\omega_c(n-m)\rho_{nm} - \frac{\kappa}{2}(n+m)\rho_{nm}$$

$$+ \kappa \sqrt{n+1}\sqrt{m+1}\, \rho_{n+1}\rho_{m+1}$$

populations: $P(n) = \rho_{nn}$

$$\Rightarrow \dot{P}(n) = -\kappa n P(n) + \kappa(n+1)P(n+1)$$



Remark: rate of decay of $P(n)$ is $n\kappa$

→ Fock states with large n are highly unstable

number of photons: $\dot{\langle \hat{n} \rangle} = \frac{d}{dt}\langle n\rangle = \mathrm{Tr}(\dot{\hat{\rho}}\, \hat{n}) = \sum_n \dot{P}(n)\, n = -\kappa \langle n\rangle$

notice ($\dot{\langle \hat{a}\rangle} = -i(\omega_c - \frac{\kappa}{2})\langle a\rangle$)

2) Phase space description

Husimi Q function: $Q(\alpha) = \frac{1}{\pi}\langle \alpha|\hat{\rho}|\alpha\rangle$

Lindblad equation $\longrightarrow$ $$\frac{\partial Q}{\partial t} = -i\omega_c\left(\alpha^*\frac{\partial Q}{\partial \alpha^*} \cdot \alpha\frac{\partial Q}{\partial \alpha}\right) + \frac{\kappa}{2}\left(\frac{\partial(\alpha Q)}{\partial \alpha} + \frac{\partial(\alpha^* Q)}{\partial \alpha^*}\right) + \kappa\frac{\partial^2 Q}{\partial\alpha\partial\alpha^*}$$

(Statistical Physics IV) Fokker-Planck equation

V - Extensions

1) Finite temperature

unitary evolution on $\mathcal{H}_S \otimes \mathcal{H}_E$

$$\hat{U}\left(|\psi\rangle \otimes |N\rangle\right) = \left(\hat{M}_0|\psi\rangle\right)\otimes|N\rangle + \left(\hat{M}_-|\psi\rangle\right)\otimes|N+1\rangle + \left(\hat{M}_+|\psi\rangle\right)\otimes|N-1\rangle$$

$\hat{M}_- \sim \sqrt{\kappa\Delta t}\sqrt{N+1}\,\hat{a}$ $\longrightarrow$ $\hat{L}_- = \sqrt{\kappa}\cdot\sqrt{N+1}\,\hat{a}$

$\lambda(\hat{a}\hat{b}^\dagger + h.c.) = \hat{H}_{S-E}$

$\hat{M}_+ \sim \sqrt{\kappa\Delta t}\sqrt{N}\,\hat{a}^\dagger$ $\qquad$ $\hat{L}_+ = \sqrt{\kappa}\cdot\sqrt{N}\,\hat{a}^\dagger$

at temperature $T = 1/\beta$ $\qquad N = \dfrac{1}{e^{\beta\omega_c}-1}$ $\qquad$ so that @ $T=0$: $\hat{L}_+ = 0$

2) Phase damping

$\hat{L} = \gamma\hat{a}^\dagger\hat{a}$ Environment "measuring the number of photons"

Fock basis: $$\dot{\rho}_{nm}(t) = \left[-i\omega_c(n-m) - \frac{\gamma}{2}(n+m)^2\right]\rho_{nm}$$

ρ_{nm} for $n \neq m \longrightarrow 0$ exponentially fast

competes with coherent drive