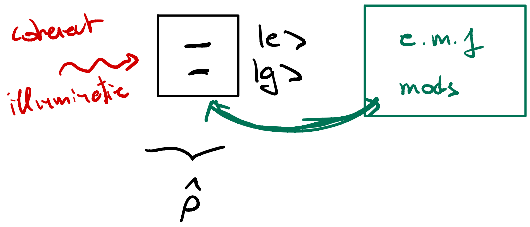
# Optical Bloch Equations

two-level system + environment



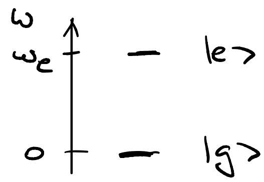

# I. Derivation

## 1) Model

$$\hat H = \Delta |e\rangle\langle e| + \frac{\Omega}{2}\overbrace{\left(|e\rangle\langle g| + |g\rangle\langle e|\right)}^{\hat\sigma_x} = \frac{\Delta}{2}\hat{1} + \vec{h}\cdot\hat{\vec{\sigma}}$$

Rk :

- $\Delta = \omega_e - \omega_{drive}$




- $\hat H$ in the frame rotating at the drive frequency + RWA

- we have chosen the phase of the external drive

  $\vec h \in xz$ plane

  in general : $\frac{\Omega}{2}\left(|e\rangle\langle g| e^{i\phi} + |g\rangle\langle e| e^{-i\phi}\right)$, $\phi$ arbitrary.

### Spontaneous emission

$$\hat L = \sqrt{\Gamma}\, |g\rangle\langle e|$$

- jumps produce transitions $|e\rangle \to |g\rangle$

- rate : $Tr(\hat\rho \hat L^\dagger \hat L) = \Gamma \cdot Tr(\hat\rho |e\rangle\langle e|) = \Gamma \cdot p_e$

## 2) Justification

Extended Hilbert space : $\mathcal{H} = \mathcal{H}_{TLS} \otimes \mathcal{H}_{em}$

Light matter interaction : $\hat H_{int} = \hat{\vec d} \otimes \hat{\vec E}$

$$\hat{\vec d} = \vec e_x \cdot d_0 \left(|e\rangle\langle g| + |g\rangle\langle e|\right) \quad \text{dipolar approximation}$$

$$\hat{\vec E} = -i \sum_{\vec k,\lambda} \sqrt{\frac{\hbar \omega_k}{2\varepsilon V}}\, \vec e_\lambda \left( e^{i(\vec k\cdot\vec r - \omega_k t)} \hat a_{\vec k,\lambda} - hc \right)$$

$\omega_k = c|\vec k|$ , $V$ quantization volume

Fermi golden rule: $$\Gamma_{\vec{k}\lambda} = \frac{2\pi}{\hbar}\left|\langle\{0\}_{\vec{k}\lambda}, e|\,\hat{\vec{d}}\otimes\hat{\vec{E}}\,|\{1_{\vec{k}\lambda}\}, g\rangle\right|^2 \cdot \delta(\omega_e - \omega_{\vec{k}})$$

Lindblad equation:
$$\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] + \sum_{\vec{k}\lambda} \hat{L}_{\vec{k}\lambda}\hat{\rho}\hat{L}^+_{\vec{k}\lambda} - \frac{1}{2}\hat{L}^+_{\vec{k}\lambda}\hat{L}_{\vec{k}\lambda}\hat{\rho} - \frac{1}{2}\hat{\rho}\hat{L}^+_{\vec{k}\lambda}\hat{L}_{\vec{k}\lambda}$$

Here! $\hat{L}_{\vec{k}\lambda} = \sqrt{\Gamma_{\vec{k}\lambda}}\,|g\rangle\langle e|$

$$\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] + \left(\sum_{\vec{k}\lambda}\Gamma_{\vec{k}\lambda}\right)|g\rangle\langle e|\hat{\rho}|e\rangle\langle g| - \frac{1}{2}|e\rangle\langle e|\hat{\rho} - \frac{1}{2}\hat{\rho}|e\rangle\langle e|$$

replace all $\hat{L}_{\vec{k}\lambda}$ by just $\hat{L} = \sqrt{\Gamma}\,|g\rangle\langle e|$, $\Gamma = \sum_{\vec{k}\lambda}\Gamma_{\vec{k}\lambda}$

Remark: • the atom is insensitive to $\vec{k},\lambda$

→ momentum is not conserved (this taken by the field)

## 3 - Bloch vector dynamics

$$\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] + \Gamma\left(|g\rangle\langle e|\hat{\rho}|e\rangle\langle g| - \frac{1}{2}\hat{\rho}|e\rangle\langle e| - \frac{1}{2}|e\rangle\langle e|\hat{\rho}\right)$$

Matrix elements:
$$\dot{\rho}_{ee} = \langle e|\dot{\hat{\rho}}|e\rangle$$
$$= -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|e\rangle - \Gamma\rho_{ee}$$
$$= -\frac{i\Omega}{2}(\rho_{ge} - \rho_{eg}) - \Gamma\rho_{ee}$$

$$\dot{\rho}_{eg} = -i\langle e|\hat{H}\hat{\rho} - \hat{\rho}\hat{H}|g\rangle - \frac{\Gamma}{2}\rho_{eg}$$
$$= -i\left(\Delta\rho_{eg} + \frac{\Omega}{2}(\rho_{gg} - \rho_{ee})\right) - \frac{\Gamma}{2}\rho_{eg}$$

$$\begin{cases} \rho_{ee} + \rho_{gg} = 1 \Rightarrow \dot{\rho}_{ee} = -\dot{\rho}_{gg} \\ \rho_{eg}^* = \rho_{ge} \end{cases}$$

$$\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{a}\cdot\hat{\vec{\sigma}}) \qquad \begin{cases} a_z = \rho_{ee} - \rho_{gg} = 2\rho_{ee} - 1 \\ a_x = \rho_{eg} - \rho_{ge} \quad \text{and} \quad a_y = i(\rho_{eg} - \rho_{ge}) \end{cases}$$

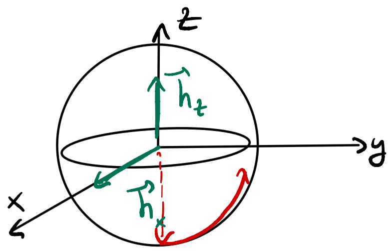
$$\boxed{\begin{cases} \dot a_z = \Omega a_y - \Gamma (a_z+1) \\ \dot a_x = -\frac{\Gamma}{2} a_x - \Delta a_y \\ \dot a_y = -\frac{\Gamma}{2} a_y + \Delta a_x - \Omega a_z \end{cases}}$$

detuning

Optical Bloch equations

external drive

choice of phase $\Longrightarrow$ appears in the $a_y$ equation




## II. Solutions

Stationary solution : $\dot{\vec a} = 0$. Let $s = \dfrac{2\Omega^2}{\Gamma^2 + 4\Delta^2}$ saturation parameter, dimensionless

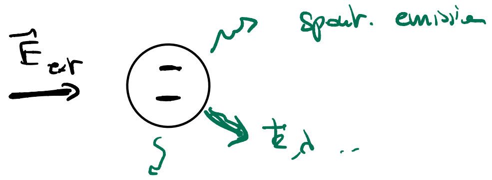
then
$$\begin{cases} a_x = -\dfrac{2\Delta}{\Omega}\,\dfrac{s}{1+s} \\ a_y = \dfrac{\Gamma}{\Omega}\,\dfrac{s}{1+s} \\ a_z = -\dfrac{1}{1+s} \end{cases}$$

Interpretation: $p_e = \langle e|\rho|e\rangle = \rho_{ee} = \dfrac{1+a_z}{2} = \dfrac{1}{2}\,\dfrac{s}{1+s}$

$\Longleftrightarrow$ rate of photon scattering :

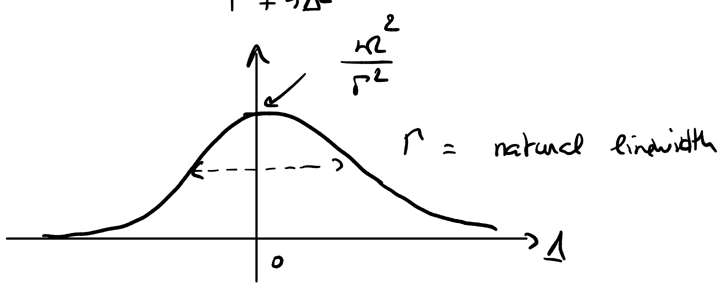
$\Gamma \rho_{ee}$ : rate of sp. emission

= photon scattering



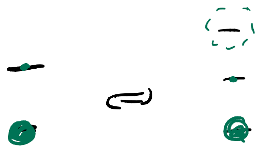

• $s \ll 1$ : low saturation → either $\Omega \ll \Gamma, \Delta$ ...
→ large $\Delta$

then $\rho_{ee} = \dfrac{s}{2} = \dfrac{\Omega^2}{\Gamma^2 + 4\Delta^2} \ll 1$ Lorentzian profile




Remark: • classical harmonic oscillator with damping rate $\Gamma$

- low $s$ $\Rightarrow$ $\rho_{ee}$



- classical:

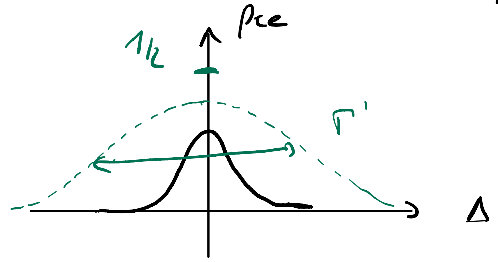
- $s \gg 1$ : strong saturation: $\rho_{ee} = \frac{1}{2}$ , $\Omega \gg \Gamma, \Delta$

  $\rightarrow$ forget $\Gamma$ : at the scale of $T_{Rabi} = \frac{2\pi}{\Omega}$, there is no spontaneous emission

  $\frac{1}{2}$ = time average of the Rabi oscillation

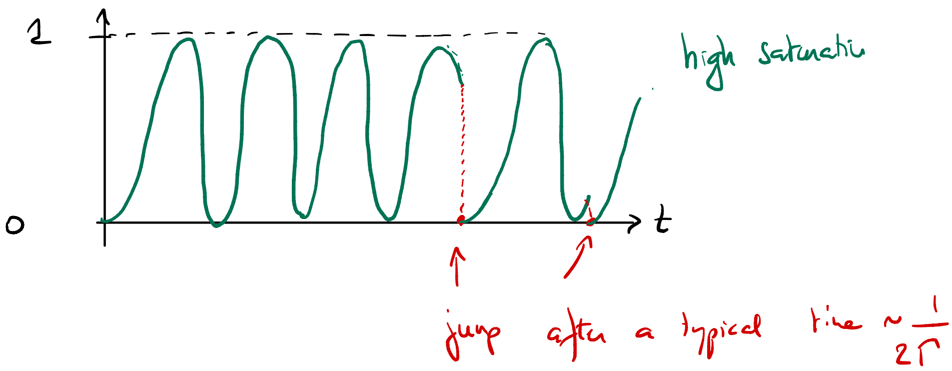
- intermediate regime: $\rho_{ee} = \dfrac{\Omega^2}{4\Delta^2 + \Gamma'^2}$ , $\Gamma' = \sqrt{\Gamma^2 + 2\Omega^2}$

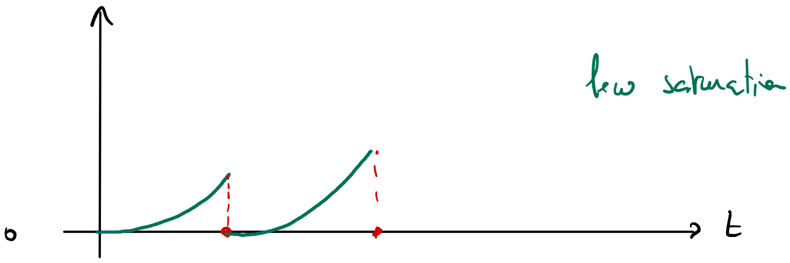
  power broadening.




## Quantum trajectory interpretation

$|\langle e|\psi(t)\rangle|^2$




Remarks: 
- we can include other processes than sp. emission

  ex: $\hat{L}_1 = \sqrt{\Gamma_1}\, |g\rangle\langle e|$

  $\hat{L}_2 = \sqrt{\Gamma_2}\, |e\rangle\langle e|$ dephasing

  then : $\dot{\rho}_{ee} = -i \langle e| \hat{H}\hat{\rho} - \hat{\rho}\hat{H} |e\rangle - \Gamma_1 \rho_{ee}$

$$\dot{\rho}_{eg} = -i \langle e | \hat{H}\hat{\rho} - \hat{\rho}\hat{H} | g \rangle - \frac{T_1}{2}\rho_{eg} - \frac{T_2}{2}\rho_{eg}$$

in <u>general</u>: $T_1$ time $\rightarrow$ decay time of $\rho_{ee}$

$T_2$ time $\rightarrow$ " " $\rho_{eg}$

Without dephasing : $T_2 = 2T_1$ $\rightarrow$ performance criteria