Lindblad Equation

idea: $\hat{\rho}(0) \rightarrow \hat{\rho}(t) = \mathcal{S}[\hat{\rho}(0)]$

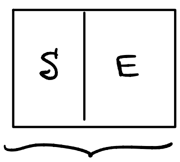
For unitary evolution we know that $i\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$

what about more general evolution?

I- Markov approximation

- there is no $1 \to 1$ correspondance between t and $t' > t$

actually:



unitary

there is a $1 \to 1$ correspondance between t and t' for $S + E$

When measurements are done on $E \rightarrow$ some information is lost

- E can also evolve in time $\rightarrow$ if E is traced out: no history of E!

  in particular $\hat{\rho}_S(t)$ may depend on $\hat{\rho}_E(t')$, for all $t' < t$

to progress: hypothesis

- E is very large: $\rightarrow$ reservoir in the thermodynamic sense

  coupling S to E does not significantly change E

- Markov approximation: memory time τ for E (property of E)

  after which any perturbation introduced by S relaxes

  $\rightarrow E$ is stationnary over timescales $\gg \tau$

"Coarse grained" description over $t \gg \tau \leftrightarrow E$ is stationnary

Examples:
- E: vacuum of electro-magnetic field in free space.
- large number of spins

  ⋮

II - Derivation of the Lindblad equation

1) Notations

- $\hat{\rho}(t+\Delta t) = \mathcal{J}[\hat{\rho}(t)] = \sum_{\mu} \hat{M}_\mu \hat{\rho}(t) \hat{M}_\mu^\dagger$

$\mathcal{J}$ match the Kro

- Markov approx : $\hat{\rho}(t+\Delta t) = \hat{\rho}(t) + \hat{O}(\Delta t)$
  - Δt small
  - $\Delta t \gg \tau$

  } coarse grained evolution

Start from Kraus:

- $\hat{M}_0 = \underbrace{\hat{\mathbb{1}}}_{\text{zeroth order}} - i \underbrace{\hat{K} \Delta t}_{1^{st}\text{ order}} + \hat{O}(\Delta t^2)$

- $\mu \geq 1$ $\quad \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger$ has to be order Δt

  So $\hat{M}_\mu = \hat{O}(\sqrt{\Delta t}) = \sqrt{\Delta t}\, \hat{L}_\mu$

- $\hat{H} = \dfrac{\hat{K} + \hat{K}^\dagger}{2}$ $\qquad \hat{J} = i\,\dfrac{\hat{K} - \hat{K}^\dagger}{2}$ such that $\hat{K} = \hat{H} - i\hat{J}$

  $\hat{H}^\dagger = \hat{H}$ $\qquad \hat{J}^\dagger = -\hat{J}$

2 - Expressions

- $\hat{M}_0 \hat{\rho} \hat{M}_0^\dagger = (\hat{\mathbb{1}} - i\hat{K}\Delta t)\hat{\rho}(\hat{\mathbb{1}} + i\hat{K}^\dagger \Delta t) + \ldots$

$= \hat{\rho} - i\Delta t(\hat{K}\hat{\rho} - \hat{\rho}\hat{K}^\dagger) + \ldots$

$= \hat{\rho} - i\Delta t\,[\hat{H},\hat{\rho}] - \Delta t\,(\hat{J}\hat{\rho} + \hat{\rho}\hat{J}) + \ldots$

Remark: for unitary evolution: only $\hat{M}_0$ in the sum, and $\hat{M}_0^\dagger \hat{M}_0 = \hat{\mathbb{1}}$

$(\hat{\mathbb{1}} + i\hat{K}^\dagger \Delta t)(\hat{\mathbb{1}} - i\hat{K}\Delta t) = \hat{\mathbb{1}} + i\Delta t\underbrace{(\hat{K}^\dagger - \hat{K})}_{=0} + \ldots$

$\Rightarrow \hat{J} = 0$

$\hat{M}_0 \hat{\rho} \hat{M}_0^\dagger = \hat{\rho} - i[\hat{H},\hat{\rho}]$

$\hat{\rho}(t+\Delta t) = \hat{\rho}(t) - i[\hat{H},\hat{\rho}] \Rightarrow i\dot{\hat{\rho}} = [\hat{H},\hat{\rho}]$

$\hat{H}$ has to be interpreted as the Hamiltonian

But it can be $\neq$ from the Hamiltonian of S alone

the difference is the Lamb-shift

For $\mu \geq 1$; $\quad \sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{1}$

$$\hat{M}_0^\dagger \hat{M}_0 + \sum_{\mu \geq 1} \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{1}$$

$$(\hat{M}_0^\dagger \hat{M}_0 =)\ \cancel{\hat{1}} - 2\Delta t \hat{J} + \Delta t \sum_{\mu \geq 1} \hat{L}_\mu^\dagger \hat{L}_\mu = \cancel{\hat{1}}$$

$$\hat{J} = \frac{1}{2} \sum_{\mu \geq 1} \hat{L}_\mu^\dagger \hat{L}_\mu$$

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{\Delta t} \cdot \left(\hat{\rho}(t+\Delta t) - \hat{\rho}(t) \right)$$

$$= \frac{1}{\Delta t} \left(\sum_\mu \hat{M}_\mu \hat{\rho}(t) \hat{M}_\mu^\dagger - \hat{\rho}(t) \right)$$

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = -i [\hat{H}, \hat{\rho}] + \sum_{\mu \geq 1} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger - \frac{1}{2} \hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_\mu^\dagger \hat{L}_\mu}$$

Lindblad equation

unitary part

Kraus operators (POVM)

$\hat{J} \to$ non unitary part of $\hat{E}$

$\to$ normalization of Kraus representation

• $\hat{L}_\mu$: jump operators, $\propto \sqrt{\mathrm{Hz}}$

III - Interpretation

• As POVM : $\{ \underbrace{\hat{M}_\mu^\dagger \hat{M}_\mu}_{\hat{E}_\mu}, \mu \}$

$$\hat{E}_0 = \hat{M}_0^\dagger \hat{M}_0 = \hat{1} - 2\Delta t \hat{J}$$

$$\hat{E}_\mu = \hat{M}_\mu^\dagger \hat{M}_\mu = \Delta t \hat{L}_\mu^\dagger \hat{L}_\mu$$

p_0 = probability to obtain 0 : $= \mathrm{Tr}(\hat{\rho} \hat{E}_0) = 1 - 2\Delta t\, \mathrm{Tr}(\hat{\rho} \hat{J})$ $\to$ no click outcome

$$= 1 - \Gamma \cdot \Delta t$$

Γ : rate

$$\langle \hat{J} \rangle = \mathrm{Tr}(\hat{\rho} \hat{J}) = \frac{\Gamma}{2}$$

p_μ = pb. to obtain μ : $= \mathrm{Tr}(\hat{E}_\mu \hat{\rho}) = \Delta t\, \mathrm{Tr}(\hat{\rho} \hat{L}_\mu^\dagger \hat{L}_\mu)$ | click on detector μ.

$$= \Delta t\, \Gamma_\mu$$

normalization : $\hat{J} = \frac{1}{2} \sum_{\mu \geq 1} \hat{L}_\mu^\dagger \hat{L}_\mu$;

- Unitary evolution over an extended Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$

$|\Psi_S\rangle \in \mathcal{H}_S$

$$\hat{U}\left(|\Psi_S\rangle \otimes |0_R\rangle\right) = \hat{M}_0|\Psi_S\rangle \otimes |0_R\rangle + \sum_{\mu \geq 1} \hat{M}_\mu |\Psi_S\rangle \otimes |\mu_R\rangle$$

$$= \left[\left(\hat{\mathbb{1}} - i\hat{H}\Delta t - \hat{J}\Delta t\right)|\Psi_S\rangle\right] \otimes |0_R\rangle + \sqrt{\Delta t}\sum_\mu \hat{L}_\mu |\Psi_S\rangle \otimes |\mu\rangle$$

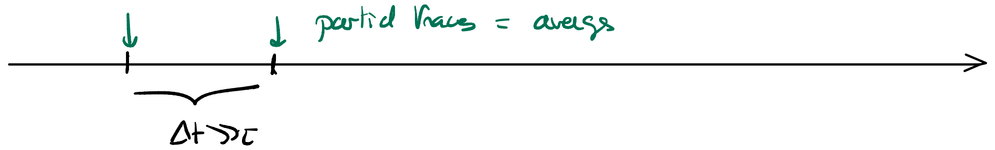
Projective measurement on the reservoir:

- with proba $p_0 \sim O(1)$ → reservoir stays in state $|0_R\rangle$
- with proba p_μ → $|\mu_R\rangle$

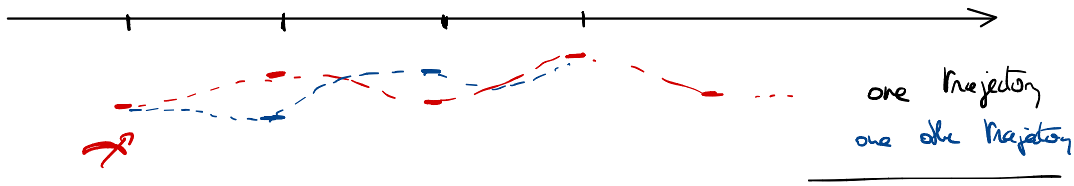
- Quantum trajectories:

  - L.E describes at each point in time the "tracing out" of the Environment.

partial trace = average



  - One realization followed in time



→ averaging over trajectories provides $\hat{\rho}(t)$

→ Monte-Carlo wavefunction method

Covered in Statistical Phys IV.