Evolution of density matrices

I. Super-operators

1) Evolution paths for $\hat{\rho}$

- Unitary evolution (closed systems) : $i\hbar\partial_t\hat{\rho} = [\hat{H},\hat{\rho}]$

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^{+}(t)$$

- Measurements :
    - with results recorded

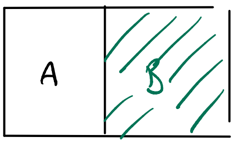
$$\hat{\rho} \longrightarrow \hat{\rho}' = \frac{1}{p(r)} \hat{M}_r\hat{\rho}\hat{M}_r^{+}$$

with $\hat{M}_r$ measurement operators $\sum_r \hat{M}_r^{+}\hat{M}_r = \hat{\mathbb{1}}$

    - without recording the result :

$$\hat{\rho}' = \sum_r \hat{M}_r\hat{\rho}\hat{M}_r^{+}$$

- Unitary evolution on extended space



$$\hat{\rho} = \hat{\rho}_A \otimes \underbrace{|0\rangle\langle 0|}_{B}$$

operator on $\mathcal{H}_A \otimes \mathcal{H}_B$
$\neq \hat{U}_A \otimes \hat{U}_B$

unitary evolution :

$$\hat{\rho}' = \hat{U}\left(\hat{\rho}_A \otimes |0\rangle\langle 0|_B\right)\hat{U}^{+}$$

partial - trace over B :

$$\hat{\rho}_A' = \mathrm{Tr}_B\,\hat{\rho}'$$

$$= \sum_\mu \langle \mu_B|\,\hat{U}\left(\hat{\rho}_A \otimes |0\rangle\langle 0|\right)\hat{U}^{+}\,|\mu_B\rangle$$

$$= \sum_\mu \underbrace{\langle \mu_B|\hat{U}|0\rangle}_{\substack{\text{operator on } \mathcal{H}_A \\ \hat{M}_\mu}} \cdot \hat{\rho}_A\, \underbrace{\langle 0|\hat{U}^{+}|\mu_B\rangle}_{\hat{M}_\mu^{+}}$$

Common feature : $\hat{\rho}' = \sum_\mu \hat{M}_\mu\hat{\rho}\hat{M}_\mu^{+}$ operator sum representation

2) Completely positive maps

$$\mathcal{S} : \hat{\rho} \to \hat{\rho}' \qquad \hat{\rho} : \text{operator on } \mathcal{H}_A$$

Requirements:

1. preserves hermiticity
2. Trace-preserving
3. Positive $\hat{\rho} \geq 0 \to \hat{\rho}' \geq 0$

3'. $\forall\ \mathcal{H}_B$ Hilbert space of system B

$\mathcal{S} \otimes \mathbb{1}$ is a positive operator | complete positivity

Completely positive map = super-operator $\hat{\mathcal{S}} : \hat{\rho} \to \hat{\rho}'$

Kraus theorem: Any completely positive map has an operator sum representation

$$\hat{\mathcal{S}}(\hat{\rho}) = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^\dagger \quad \text{for a set of } \{\hat{M}_\mu\} \text{ such that}$$

$$\sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{1}}$$

Remark:

- if $\dim(\mathcal{H}_A) = N$ then there are at most N^2 operators
- the representation is not unique

$$\hat{N}_\nu = \sum_\mu U_{\nu\mu} \hat{M}_\mu \qquad U : \text{unitary matrix}$$

- Any super-operator can be interpreted as a POVM
- In general, a super operator is NOT invertible → Decoherence

  the only case where S.O is invertible is unitary evolution

II - Quantum Channels

$$\hat{\rho} \longrightarrow \boxed{} \longrightarrow \hat{\rho}'$$

1) Amplitude damping channel

model for spontaneous emission

System S (qbit)

Environment E

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$$

unitary evolution:

$p \in [0,1]$

$$|0\rangle \otimes |0\rangle \longrightarrow |0\rangle \otimes |0\rangle$$

$$|1\rangle \otimes |0\rangle \longrightarrow \sqrt{p}\,|0\rangle \otimes |1\rangle + \sqrt{1-p}\,|1\rangle \otimes |0\rangle$$

$$\hat{\mathcal{S}} : \hat{\rho}_S \rightarrow \hat{\rho}_S' = \mathrm{Tr}_E\left[\hat{U}\hat{\rho}\hat{U}^+\right]$$

initialize: $\hat{\rho} = \hat{\rho}_S \otimes |0\rangle\langle 0|_E$

$$\hat{\rho}' = \langle 0_E|\hat{U}\hat{\rho}\hat{U}^+|0_E\rangle + \langle 1_E|\hat{U}\hat{\rho}\hat{U}^+|1_E\rangle$$

$$= \hat{M}_0\,\hat{\rho}_S\,\hat{M}_0^+ + \hat{M}_1\hat{\rho}_S\hat{M}_1^+$$

$\hat{\rho}_S$, $\hat{M}_0$, $\hat{M}_1$ operators on $\hat{\mathcal{H}}_S$

- $\underline{\hat{M}_0 = ?}$: $\underbrace{\langle 0_E|\hat{U}\left(\hat{\rho}_S \otimes |0\rangle\langle 0|_E\right)\hat{U}^+|0_E\rangle}$ $= \hat{M}_0\hat{\rho}_S\hat{M}_0^+$

$$\hat{M}_0 = \langle 0_E|\hat{U}|0_E\rangle$$

$$= \begin{pmatrix} \underset{M_{0,00}}{\langle 0_S|\otimes\langle 0_E|\hat{U}|0_S\rangle\otimes|0_E\rangle} & \underset{M_{0,01}}{\langle 0_S|\otimes\langle 0_E|\hat{U}|1_S\rangle\otimes|0_E\rangle} \\ \underset{M_{0,10}}{\langle 1_S|\otimes\langle 0_E|\hat{U}|0_S\rangle\otimes|0_E\rangle} & \underset{M_{0,11}}{\langle 1_S|\otimes\langle 0_E|\hat{U}|1_S\rangle\otimes|0_E\rangle} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

$\underline{\hat{M}_1 = ?}$ $\langle 1_E|\hat{U}\left(\hat{\rho}_S \otimes |0_E\rangle\langle 0|\right)\hat{U}^+|1_E\rangle$

$$\hat{M}_1 = \langle 1_E|\hat{U}|0_E\rangle$$

$$= \begin{pmatrix} \langle 0_S| \otimes \langle 1_E| \hat{U} |0_S\rangle \otimes |0_E\rangle & \langle 0_S| \otimes \langle 1_E| \hat{U} |1_S\rangle \otimes |0_E\rangle \\ \langle 1_S| \otimes \langle 1_E| \hat{U} |0_S\rangle \otimes |0_E\rangle & \langle 1_S| \otimes \langle 1_E| \hat{U} |1_S\rangle \otimes |0_E\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

Check: $\hat{\Pi}_0^\dagger \hat{\Pi}_0 + \hat{\Pi}_1^\dagger \hat{\Pi}_1 = \hat{\mathbb{1}}$

$$\hat{\rho}_S = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \quad \text{then} \quad \hat{\rho}_S' = \begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\cdot\rho_{10} & \rho_{11}(1-p) \end{pmatrix}$$

- probability of transition p between 0 and 1
- $\sqrt{1-p}$ reduction in the coherence!

Bloch sphere: $\hat{\rho}_S = \frac{1}{2}(\hat{\mathbb{1}} + \vec{a}\cdot\vec{\sigma})$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \longrightarrow \vec{a}' = \begin{pmatrix} \sqrt{1-p}\, a_x \\ \sqrt{1-p}\, a_y \\ (1-p) a_z + p \end{pmatrix}$$

$$\mathrm{Tr}(\rho_S'^2) \leq \mathrm{Tr}(\rho_S^2) \quad \forall p \in [0,1]$$

Interpretation in terms of measurements

$$\hat{E}_0 = \hat{\Pi}_0^\dagger \hat{\Pi}_0 \qquad \hat{E}_1 = \hat{\Pi}_1^\dagger \hat{\Pi}_1$$

- $\hat{\Pi}_1$: $|\psi\rangle \rightarrow \dfrac{\hat{\Pi}_1|\psi\rangle}{\sqrt{p}}$ $\qquad \hat{\Pi}_1 = \sqrt{p}\,|0\rangle\langle 1|$

  "1 click" $\qquad \hat{E}_1 = p\,|1\rangle\langle 1|$

  $|\psi\rangle \rightarrow |0\rangle$ (Heralded preparation of $|0\rangle$)

- $\hat{\Pi}_0$: $|\psi\rangle \rightarrow \dfrac{\hat{\Pi}_0|\psi\rangle}{\sqrt{p(0)}}$ $\qquad \hat{\Pi}_0 = |0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|$

  $\qquad \hat{E}_0 = |0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$

$$|\Psi\rangle = a|0\rangle + b|1\rangle$$

$$\hat{M}_0 |\Psi\rangle \propto a|0\rangle + b\sqrt{1-p}\,|1\rangle \quad (\text{up to normalization})$$

"0 click"

No click $\neq$ No information

<u>Remark:</u> repeat n consecutive A.D channels: $\rho_{11} \to \rho_{11}(1-p)^n$

decay rate Γ $\quad p = \frac{\Gamma t}{n}$ $\quad \rho_{11} \to \rho_{11} e^{-\Gamma t} \quad (n\to\infty)$

2) Phase damping channel

unitary operation onto $\mathcal{H}_S \otimes \mathcal{H}_E$ $\qquad p \in [0,1]$

$$\begin{cases} |0\rangle\otimes|0\rangle \to \sqrt{p}\,|0\rangle\otimes|0\rangle + \sqrt{1-p}\,|0\rangle\otimes|1\rangle \\ |1\rangle\otimes|0\rangle \to \sqrt{p}\,|1\rangle\otimes|0\rangle - \sqrt{1-p}\,|1\rangle\otimes|1\rangle \end{cases}$$

$$\iff \begin{cases} |0\rangle\otimes|0\rangle \to |0\rangle\otimes\left(\sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle\right) \\ |1\rangle\otimes|0\rangle \to |1\rangle\otimes\left(\sqrt{p}|0\rangle - \sqrt{1-p}|1\rangle\right) \end{cases}$$

<u>Remark:</u> it is an entangling operation: $\frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes|0\rangle \to \dots$

$$\hat{\rho}_S' = \mathrm{Tr}_E\left[\hat{U}\,\overbrace{\left(\hat{\rho}_S\otimes|0\rangle\langle 0|_E\right)}^{\hat{\rho}}\,\hat{U}^+\right]$$

$$= \langle 0_E|\hat{U}\hat{\rho}\hat{U}^+|0_E\rangle + \langle 1_E|\hat{U}\hat{\rho}\hat{U}^+|1_E\rangle$$

$$= \hat{M}_0\hat{\rho}_S\hat{M}_0^+ + \hat{M}_1\hat{\rho}_S\hat{M}_1^+$$

$$\hat{M}_0 = \sqrt{p}\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad \hat{M}_1 = \sqrt{1-p}\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}}_{\hat{\sigma}_z} = \sqrt{1-p}\,\hat{\sigma}_z$$

check that $\hat{M}_0^+\hat{M}_0 + \hat{M}_1^+\hat{M}_1 = \hat{\mathbb{1}}$

<u>Bloch vector:</u> $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{1}} + \vec{a}\cdot\vec{\sigma})$ $\quad \begin{pmatrix}a_x\\a_y\\a_z\end{pmatrix} = \vec{a} \to \vec{a}' = \begin{pmatrix}a_x(2p-1)\\a_y(2p-1)\\a_z\end{pmatrix}$

unless $p=0$ or 1 we have $\|\vec{a}'\|^2 < \|\vec{a}\|^2$

<u>interpretation:</u> $\frac{|0\rangle+|1\rangle}{\sqrt{2}}\otimes|0\rangle = \sqrt{p}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes|0\rangle + \sqrt{1-p}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\otimes|1\rangle$

if $p = \frac{1}{2}$: $\hat{\rho}' = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

Remarks

- irreversible: not possible to "inflate the Bloch sphere"
- generalize to harmonic oscillator
- Physical motivation: random rotations about z

$$\hat{R}_z(\theta) = e^{i\theta\hat{\sigma}_z}$$

average over possible values of θ: $\hat{\rho}' = \frac{1}{\sqrt{4\pi\lambda}} \int d\theta\, \hat{R}_z(\theta)\,\hat{\rho}\,\hat{R}_z(\theta)^{+} e^{-\theta^2/4\lambda}$

$|\psi\rangle = a|0\rangle + b|1\rangle$:

$$\hat{\rho}' = \begin{pmatrix} |a|^2 & ab^* e^{-\lambda} \\ a^* b e^{-\lambda} & |b|^2 \end{pmatrix}$$

3) Depolarizing channel:

Phase damping picks a preferred direction : z

$$\hat{M}_0 = \sqrt{1-p}\,\hat{1} + \sqrt{p}\,\hat{\sigma}_z$$

Now : define a channel through Kraus operators:

$$\hat{M}_0 = \sqrt{1-p}\,\hat{1}$$

$$\hat{M}_1 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_x \qquad \hat{M}_2 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_y \qquad \hat{M}_3 = \sqrt{\frac{p}{3}}\,\hat{\sigma}_z$$

$$\sum_i \hat{M}_i^{+}\hat{M}_i = \hat{1}$$

Action on the Bloch vector:

$$\hat{\rho}' = \sum_\mu \hat{M}_\mu \hat{\rho} \hat{M}_\mu^{+} = \frac{1}{2}\left(\hat{1} + \vec{a}\cdot\begin{pmatrix} \sum_\mu \hat{M}_\mu \hat{\sigma}_x \hat{M}_\mu^{+} \\ \vdots \end{pmatrix}\right)$$

$$\sum_\mu \hat{M}_\mu \hat{\sigma}_x \hat{M}_\mu^{+} = \dots = \left(1 - \frac{4p}{3}\right)\hat{\sigma}_x$$, same for $\hat{\sigma}_y$, $\hat{\sigma}_z$

so: $\vec{a}' = \vec{a}\cdot\left(1 - \frac{4p}{3}\right)$ uniform "contraction" of the Bloch sphere