Quantum Measurements

I - Projective measurements

Let $\hat{M}$ be an observable $\hat{M} = \sum_m m \hat{P}_m$

where $\hat{P}_m$: projector onto the eigenspace of m

- Measurement: upon observing outcome m, the state of the system evolve:

$$|\psi\rangle \rightarrow \frac{\hat{P}_m|\psi\rangle}{\sqrt{p(m)}} = |\psi_m\rangle$$

and the probability to observe m is $p(m) = \langle\psi|\hat{P}_m|\psi\rangle$

- For a mixed state: $\hat{\rho}$: $\underline{p}(m) = \underline{\mathrm{Tr}\ (\hat{\rho}\hat{P}_m)} = E[M] = \langle\hat{P}_m\rangle$

2 averages: . one quantum average

. classical probabilistic average

if m has been observed (as outcome of the measurement):

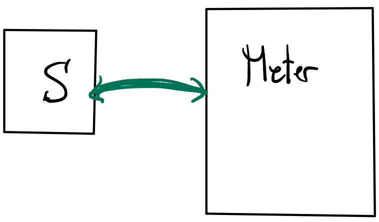
$$\hat{\rho}_m = \frac{\hat{P}_m \hat{\rho} \hat{P}_m^+}{p(m)}$$

if measurement is performed but the outcome is not recorded:

$$\hat{\rho} \rightarrow \sum_m \hat{\rho}_m\, p(m) = \sum_m \hat{P}_m \hat{\rho} \hat{P}_m^+$$

if $|\psi\rangle\langle\psi| \longrightarrow \sum_m p(m)\, \hat{P}_m|\psi\rangle\langle\psi|\hat{P}_m^+ = \sum_m p(m)\, |\psi_m\rangle\langle\psi_m|$

II - System-meter formulation



projective measurement

2 Steps: (1): coupling between S and M

$$|\psi_{SM}\rangle \in \mathcal{H}_M \otimes \mathcal{H}_S$$

$|\Psi_{sm}\rangle = |\theta\rangle \otimes |\varphi\rangle \qquad |\theta\rangle \in \mathcal{H}_m, \quad |\varphi\rangle \in \mathcal{H}_s$

$|\Psi_{sm}\rangle \longrightarrow |\Psi'\rangle = \hat{U}\,|\Psi_{sm}\rangle \neq |\theta'\rangle \otimes |\varphi'\rangle$

② : Projective measurement of Meter: let r the outcome

$$\hat{P}_r = \hat{\Pi}_r \otimes \hat{\mathbb{1}}$$

$$|\Psi'_r\rangle = \frac{\hat{P}_r|\Psi'\rangle}{\sqrt{p(r)}} = \frac{(\hat{\Pi}_r \otimes \hat{\mathbb{1}})\,|\Psi'\rangle}{\sqrt{p(r)}}$$

$$= \frac{(\hat{\Pi}_r \otimes \hat{\mathbb{1}})\,\hat{U}\,|\theta\rangle \otimes |\varphi\rangle}{\sqrt{p(r)}}$$

Suppose $\hat{\Pi}_r = |r\rangle\langle r|$:

operator onto $\mathcal{H}_s$

$$|\Psi'_r\rangle = \frac{|r\rangle \otimes \langle r|\hat{U}|\theta\rangle\,|\varphi\rangle}{\sqrt{p(r)}}$$

Let $\hat{M}_r = \langle r|\hat{U}|\theta\rangle$: operator on $\mathcal{H}_s$

So after the 2 steps: $|\Psi'_r\rangle = \dfrac{|r\rangle \otimes \hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$

Consequences:

- start from separable state $|\theta\rangle \otimes |\varphi\rangle$

  ↳ end with a separable state : $|\Psi'_r\rangle$

  $$|\varphi\rangle \longrightarrow \frac{\hat{M}_r|\varphi\rangle}{\sqrt{p(r)}}$$

- $\hat{M}_r$ is not a projector : "measurement operator"

- $p(r) = \langle \Psi'|\hat{P}_r|\Psi'\rangle = \langle\varphi|\hat{M}_r^{+}\hat{M}_r|\varphi\rangle$

III - Generalized measurements

1. Definition

Let $\hat{M}_r$ measurement operators, and $\hat{E}_r = \hat{M}_r^{+}\hat{M}_r$, such that

- $\sum_r \hat{E}_r = \hat{\mathbb{1}}$: $\hat{E}_r$ are probability operators
- $\hat{E}_r$: hermitian, positive
- measurement outcomes are labelled by r

$\{\hat{E}_r, r\}$: Positive operator-valued measure (POVM)

Then : $p(r)$: probability of outcome r

$$p(r) = \mathrm{Tr}\,(\hat{\rho}\,\hat{E}_r)$$

After the measurement : • if r has been observed $\hat{\rho} \to \hat{\rho}_r = \dfrac{\hat{M}_r \hat{\rho} \hat{M}_r^{+}}{p(r)}$

• if the outcome is not recorded : $\hat{\rho} \to \sum_r \hat{\rho}_r \cdot p(r)$

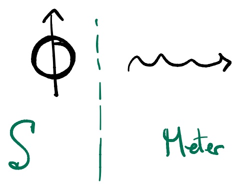
$$= \sum_r \hat{M}_r \hat{\rho} \hat{M}_r^{+}$$

Remarks : • it is not the most general formulation

• given $\hat{E}_r$ positive hermitian operator $\to \hat{M}_r = \sqrt{\hat{E}_r}$

2) Examples

Photon counting + spontaneous emission



$|\Psi_{SM}\rangle$: pure state $\in \mathcal{H}_S \otimes \mathcal{H}_M$

Measurement of the number of photons : j photons : $\hat{M}_j = \hat{1} \otimes |0\rangle\langle j|$

$\hat{E}_j = \hat{M}_j^{+} \hat{M}_j = \hat{1} \otimes |j\rangle\langle j|$, positive operator

$\sum_j \hat{E}_j = \hat{1}$ $\qquad \{ \hat{E}_j, j \}$ POVM !

. After the measurement : $|\Psi_{SM,j}\rangle = \dfrac{|\Psi_S\rangle \otimes |0\rangle}{\sqrt{p(j)}}$ (pure) product state

. if the result is not recorded : $|\Psi_{SM}\rangle\langle\Psi_{SM}| \longrightarrow \sum_j \hat{M}_j |\Psi_{SM}\rangle\langle\Psi_{SM}| \hat{M}_j^{+}$

$$|\Psi_{SM}\rangle = \sum_{k\ell} \alpha_{k\ell}\, |k\rangle \otimes |\ell\rangle$$

$$|\Psi_{SM}\rangle\langle\Psi_{SM}| \to \sum_j (\hat{1} \otimes |0\rangle\langle j|) \sum_{\substack{k\ell \\ k'\ell'}} \alpha_{k\ell} \,(|k\rangle \otimes |\ell\rangle)(\langle k'| \otimes \langle \ell'|)\, \alpha^*_{k'\ell'} \, . \sum_j (\hat{1} \otimes |j\rangle\langle 0|)$$

$$= \underbrace{\sum_{kk'} \Big(\sum_j \alpha^*_{k'j}\, \alpha_{kj} \Big)}\, |k\rangle\langle k'| \otimes |0\rangle\langle 0|$$

$$= \mathrm{Tr}_M \left(|\Psi_{sn}\rangle\langle\Psi_{sn}| \right) \otimes |0\rangle\langle$$