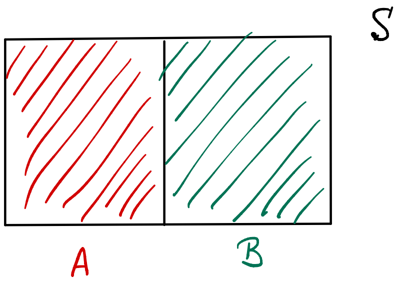
Bipartite systems - Entanglement

I. Density matrix



1) Bipartite system

$$\mathcal{H}_S = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$|\Psi_{AB}\rangle \in \mathcal{H}_S$$

Let $\hat{M}_A$: observable on system A

$\hat{M} = \hat{M}_A \otimes \hat{\mathbb{1}}$: observable on S (acts on A only)

$$\langle \hat{M} \rangle = \langle \Psi_{AB} | \hat{M} | \Psi_{AB} \rangle$$

now write $|\Psi_{AB}\rangle = \sum_{i\mu} \alpha_{i\mu} |i_A\rangle \otimes |\mu_B\rangle$

$$= \sum_{\substack{i\mu \\ j\nu}} \langle i_A | \hat{M}_A | j_A \rangle \cdot \langle \mu_B | \hat{\mathbb{1}} | \nu_B \rangle \, \alpha^*_{i\mu} \alpha_{j\nu}$$

$$= \sum_{ij\mu} \alpha_{i\mu}^* \alpha_{j\mu} \langle i | \hat{M}_A | j \rangle$$

$$\hat{\rho}_A = \sum_{\mu, j, i} \alpha^*_{i\mu} \alpha_{j\mu} |j\rangle\langle i| \quad \text{density matrix of } A$$

$$\langle \hat{M} \rangle = \mathrm{Tr}\left(\hat{\rho}_A \hat{M}_A\right)$$

- $\hat{\rho}_A$ is obtained from $|\Psi_{AB}\rangle$: partial trace $\hat{\rho}_A = \mathrm{Tr}_B\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right]$

2) States:

- Pure state on A: $|\Psi_A\rangle \in \mathcal{H}_A$ such that $\hat{\rho}_A = |\Psi_A\rangle\langle\Psi_A|$

- Product state | Separate of S : $|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$ for some $|\Psi_A\rangle$ and $|\Psi_B\rangle$

- A state which is not a product state is entangled

  $\Leftrightarrow$ A is entangled with B when S is in state $|\Psi_{AB}\rangle$

3) Discussion

- here we defined entanglement for bipartite systems. ⚠ more than 2 parts

- $\hat{\rho}_A$ can be diagonalized $\hat{\rho}_A = \sum_i p_i |i\rangle\langle i|$ $\quad p_i$: probability for A to be in state $|i\rangle$

$\hat\rho_A$: represents an ensemble of states.

$$\langle \hat{M}_A\rangle = \mathrm{Tr}(\hat\rho_A \hat M_A) \longrightarrow$$

(1) : q-mechanical expectation value for each state of the ensemble $\langle i|\hat M_A|i\rangle$

(2) : classical average over all possible states $|i\rangle$

- Purity : $\hat\rho_A^{\,2} = \sum_i p_i^2 |i\rangle\langle i|$ So $\mathrm{Tr}(\hat\rho_A^{\,2}) = \sum_i p_i^2 \leq \sum_i p_i = 1$

II – Entanglement

Bipartite system $\hat\rho_A = |\Psi_A\rangle\langle\Psi_A|$

$$|\Psi_{AB}\rangle = |\Psi_A\rangle\otimes|\Psi_B\rangle$$

$$\hat\rho_A = \mathrm{Tr}_B\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right] = \sum_\mu |\Psi_A\rangle\cdot\langle\mu_B|\Psi_B\rangle\cdot\langle\Psi_A|\langle\Psi_B|\mu_B\rangle$$

$$= |\Psi_A\rangle\cdot\underbrace{\sum_\mu\left|\langle\mu_B|\Psi_B\rangle\right|^2}_{1}$$

if $\hat\rho_A$ is mixed : A and B were entangled

1) Entropy

given $\hat\rho_A$: Renyi entropy $S_n(\hat\rho_A) = -\mathrm{Log}\left[\mathrm{Tr}(\hat\rho_A^{\,n})\right]\frac{1}{n-1}$

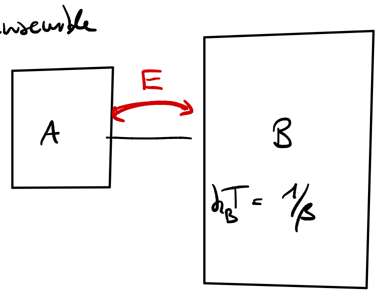
Entanglement entropy $S(\hat\rho_A) = -\mathrm{Tr}\left[\hat\rho_A \ln\hat\rho_A\right]$

$$= \lim_{n\to1} S_n(\hat\rho_A)$$

$$S\left(|\Psi\rangle\langle\Psi|\right) = 0$$

Ex: Statistical mechanics , canonical ensemble

$$\hat\rho_A = \frac{1}{Z}e^{-\beta\hat H_A}$$



$$S[\hat\rho_A] = \dots = -\beta F + \beta\langle\hat H_A\rangle$$

F : free energy $= \left(e^{-\beta F} = Z\right)$

Remarks :

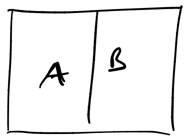
- For any bipartition of S into A/B : $S(\hat\rho_A) = S(\hat\rho_B)$
- Maximum entropy : $\hat\rho_A = \frac{1}{N}\sum_i |i_A\rangle\langle i_A|$ $N = \dim(\mathcal{H}_A)$

$$S(\hat{\rho}_A) = \operatorname{Log} N$$

$$\hat{\rho}_A = \frac{1}{N} \sum_i |i_A\rangle\langle i_A| \quad \text{infinite temperature}$$

- Thermalization: $\Rightarrow$ building up entanglement.

S



$S \propto$ area of the boundary A/B
volume of A

- Connection between S and quantum gravity

2) Schmidt decomposition

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\Psi\rangle \in \mathcal{H}_{AB}$

there exists an orthonormal set in $\mathcal{H}_A$: $\{|i_A\rangle\}$

an " " " $\mathcal{H}_B$: $\{|i_B\rangle\}$

Such that: $$|\Psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

$\{\lambda_i\}$: Schmidt coef. $\quad \#\{\lambda_i\}$ = Schmidt rank

Density matrix: $\hat{\rho}_A = \operatorname{Tr}_B\left[|\Psi\rangle\langle\Psi|\right]$ Let $\{|i_B\rangle\}$ the "Schmidt basis"

$$= \sum_i \langle i_B|\Psi\rangle\langle\Psi|i_B\rangle$$

$$= \sum_i |\lambda_i|^2 |i_A\rangle\langle i_A|$$

$$S(\hat{\rho}_A) = -\sum_i |\lambda_i|^2 \operatorname{Log} \lambda_i^2$$

Remark:

- $\lambda_i = e^{-\varepsilon_i/2}$ $\quad |\Psi\rangle = \sum_i e^{-\varepsilon_i/2} |i_A\rangle \otimes |i_B\rangle$

  $\{\varepsilon_i\}$: entanglement spectrum

- given $\hat{\rho}_A = \sum_i p_i |i_A\rangle\langle i_A|$

  there exists R, with Hilbert space $\mathcal{H}_R$ such that:

  $$\exists\, |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R \quad \hat{\rho}_A = \operatorname{Tr}_R\left[|\Psi\rangle\langle\Psi|\right]$$

  purification of $\hat{\rho}_A$

3) <u>Example:</u> 2 qbit system: A: 1qbit B: 1qbit

$\mathcal{H}_A = \mathrm{Sp}\{|0\rangle, |1\rangle\}$ $\mathcal{H}_B = \mathrm{Sp}\{|0\rangle, |1\rangle\}$

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\mathcal{H} = \mathrm{Sp}\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ $\left(|ij\rangle = |i\rangle_A \otimes |j\rangle_B \right)$

- $\frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$ → not entangled: $\frac{1}{\sqrt{2}} |0\rangle \otimes \left(|0\rangle + |1\rangle\right)$

- $\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ → yes!
$= |\phi^+\rangle$ $\mathrm{Tr}_B\left[|\phi^+\rangle\langle\phi^+|\right] = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$ maximally mixed state

Bell state

- $|\phi_\varepsilon\rangle = \frac{1}{\sqrt{\varepsilon^2 + (1-\varepsilon)^2}} \cdot \left(\varepsilon|00\rangle + (1-\varepsilon)|11\rangle\right)$ → yes!

$S = \ldots \sim \varepsilon^2$

- $|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ $|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$ Bell states
$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$ $|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$