Two level systems

I – States and operators

- Hilbert space $\mathcal{H} = \mathrm{Sp}\{|\uparrow\rangle, |\downarrow\rangle\}$, or $|g\rangle, |e\rangle$ or $|0\rangle, |1\rangle$

  State $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

  one possible parametrization : $|\psi\rangle = \cos\theta/2\,|\uparrow\rangle + e^{i\phi}\sin\theta/2\,|\downarrow\rangle$

- Hermitian operators on $\mathcal{H}$: spanned by $\hat{\mathbb{1}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$

$\hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\hat{\sigma}_+ = |\uparrow\rangle\langle\downarrow| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \hat{\sigma}_- = |\downarrow\rangle\langle\uparrow| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{\sigma}_+ = \frac{\hat{\sigma}_x + i\hat{\sigma}_y}{2} \qquad \hat{\sigma}_- = \frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}$$

Properties of Pauli matrices: $\quad \mathrm{Tr}\,\hat{\sigma}_i = 0 \qquad i = x, y, z$

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{\mathbb{1}} \Rightarrow \hat{\sigma}_i^2 = \hat{\mathbb{1}}$$

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z \quad + \text{ circ permutations}$$

Notation: $\hat{\vec{\sigma}} = \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}$

Representation of operators: $\hat{O} = \frac{1}{2}\left\{ \hat{\mathbb{1}}\,\mathrm{Tr}(\hat{O}) + \sum_i \mathrm{Tr}(\hat{O}\hat{\sigma}_i)\cdot\hat{\sigma}_i \right\}$

{ Think about: $A = \sum_i \langle A, u_i\rangle u_i$

notation: $\hat{O} = \frac{1}{2}\left\{ \hat{\mathbb{1}}\,\mathrm{Tr}(\hat{O}) + \vec{n}_0\cdot\hat{\vec{\sigma}} \right\}$ $\qquad \vec{n}_0 = \begin{pmatrix} n_{0x} \\ n_{0y} \\ n_{0z} \end{pmatrix}$

II - Bloch sphere

1) Density matrix

$\hat{\rho}$: $Tr(\hat{\rho}) = 1$ and $\hat{\rho} = \frac{1}{2}\{\hat{\mathbb{1}} + \vec{n}\cdot\vec{\sigma}\}$

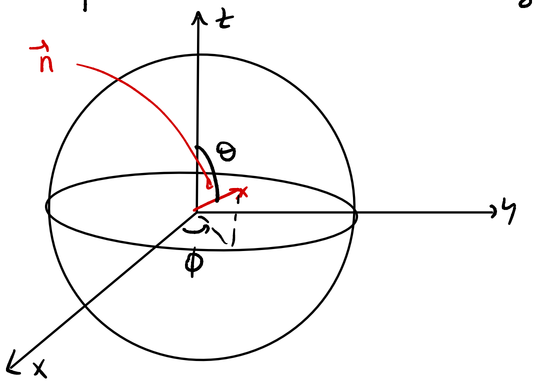
$\vec{n}$: Bloch vector

$Tr(\hat{\rho}^2) \leqslant 1$ with $Tr(\hat{\rho}^2) = 1 \Rightarrow \hat{\rho} = |\psi\rangle\langle\psi|$

$$Tr(\hat{\rho}^2) = Tr\left\{\frac{1}{4}(\hat{\mathbb{1}} + \vec{n}\cdot\vec{\sigma})(\hat{\mathbb{1}} + \vec{n}\cdot\vec{\sigma})\right\}$$

$$= \frac{1 + \|\vec{n}\|^2}{2}$$

pure stats are characterized by $\|\vec{n}\|^2 = 1$



$$\vec{n} = \|\vec{n}\| \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

pure stats : $\|\vec{n}\| = 1$

$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\theta/2\,|\downarrow\rangle$

$\hat{\rho} = |\psi\rangle\langle\psi|$ $\quad Tr(\hat{\rho}\,\hat{\sigma}_x) = \ldots$

- Examples:
  - $|\uparrow\rangle$: north pole
  - $|\downarrow\rangle$: south pole
  - equal weight superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$:

    $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + e^{i\phi}|\downarrow\rangle)$

    equator of the BS
  - equal weight incoherent mixture: $\hat{\rho} = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$
  - general incoherent mixture: $\hat{\rho} = p_e|\uparrow\rangle\langle\uparrow| + (1-p_e)|\downarrow\rangle\langle\downarrow|$

    $= \frac{1}{2}(\hat{\mathbb{1}} + (2p_e - 1)\hat{\sigma}_z)$

2) Hamiltonian

$$\hat{H} = E_0\hat{\mathbb{1}} - \vec{h}\cdot\vec{\sigma}$$

Ex : spin ½ in a $\vec{B}$ field , $\hat{H} = -\vec{\hat{\mu}}\cdot\vec{B}$ $\quad \vec{\hat{\mu}} = \mu\,\vec{\hat{\sigma}}$

Dynamics: $i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}]$ Liouville equation

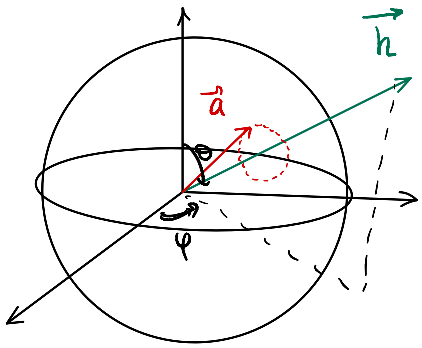
$\Big\{$ Rk: $\neq$ Heisenberg picture $i\hbar \partial_t \hat{O} = [\hat{O}, \hat{H}]$

E_0 does not generate any dynamics $\hat{\rho}(t) = \frac{1}{2}\left(\hat{\mathbb{1}} + \vec{a}(t)\cdot\vec{\sigma}\right)$

$i\hbar \dot{\hat{\rho}} = \dots$

$\dot{\vec{a}} = \frac{2\vec{h}}{\hbar} \times \vec{a}$ precession of $\vec{a}$ about the vector $\vec{h}$

frequency : $\frac{2\|\vec{h}\|}{\hbar}$



<u>Eigenstates of $\hat{H}$</u> :

if θ, φ are the coordinates of $\vec{h}$

then $|\psi_+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle$

$|\psi_-\rangle = \sin\frac{\theta}{2}|\uparrow\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow\rangle$

Rotations on the Bloch sphere:

$e^{-i\hat{H}/\hbar\, t}$ rotation by angle $\frac{2\|\vec{h}\|}{\hbar}\cdot t$ around axis $\vec{h}$

$$\hat{R}_{\vec{n}}(\theta) = e^{-\frac{i\theta}{2}\vec{n}\cdot\vec{\sigma}}$$

Ex: Rabi oscillations: $\vec{h}$ along x $\rightarrow$ any $\vec{h}$ in the xy plane