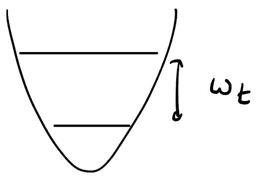
Sideband Cooling

So far : free space : $|\vec{k}\rangle$, $\vec{k}$ was continuously distributed

variations of $\vec{k}$ are continuous $\Rightarrow$ Force

Discrete spectrum:



harmonic trap

Physical realizations:

- atoms / ions in traps
- mechanical oscillators

I – Motional sidebands

- Hilbert space : $\mathcal{H} = \mathcal{H}_{com} \otimes \mathcal{H}_{internal}$

  Field : driven mode $\rightarrow$ coherent state (laser)

  vacuum considered as a reservoir

  Two-level atom : $\mathcal{H}_{int} = Sp\{|g\rangle, |e\rangle\}$

  $\mathcal{H}_{com} = Sp\{|n\rangle, n \in \mathbb{N}\}$ eigenstates of the harmonic trap

  (only one dimension)

- Hamiltonian:

$$\hat{H} = \omega_e |e\rangle\langle e| \otimes \hat{\mathbb{1}} + \omega_t \hat{\mathbb{1}} \otimes \hat{a}^\dagger \hat{a} + \frac{\Omega}{2}\left(|e\rangle\langle g| \otimes e^{i(k\hat{z} - \omega_L t)} + h.c. \right)$$

operator on $\mathcal{H}_{com}$

$\hat{a}$: annihilates vibrational quanta ("phonons")

1) Lamb-Dicke parameter

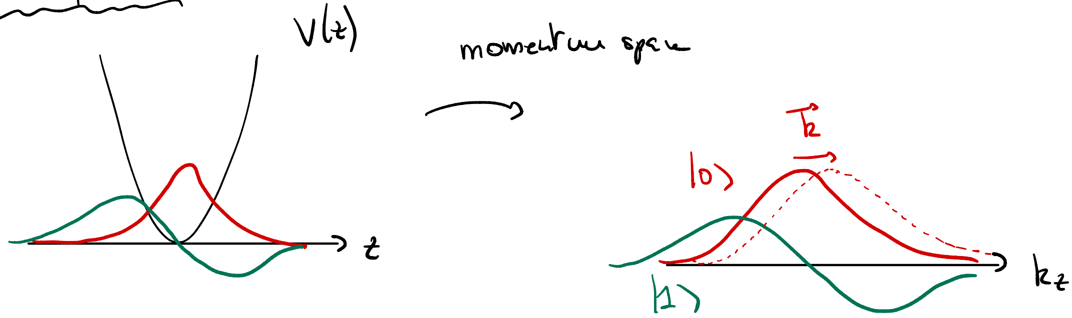
$$\hat{z} = z_0 (\hat{a} + \hat{a}^\dagger) \quad \text{with } z_0 = \sqrt{\frac{\hbar}{m\omega_t}} \quad \text{harmonic length}$$

$$e^{ik\hat{z}} = e^{ikz_0(\hat{a} + \hat{a}^\dagger)}$$

$$\boxed{\eta = kz_0}$$

Lamb-Dicke parameter

Physical interpretation:



photon absorption : $|\vec{p}\rangle \longrightarrow |\vec{p}+\vec{k}\rangle$ translation in k space

η measures this displacement w.r to the size of the wavefunct.

2) Sideband spectrum

$$e^{i\underbrace{kz_0}_{\eta}(\hat{a}+\hat{a}^\dagger)} = e^{i\eta\hat{a}^\dagger} e^{i\eta\hat{a}} e^{-\eta^2/2}$$

$$= \sum_{n,m} \frac{(i\eta)^{n+m}}{n!\,m!} \hat{a}^{\dagger m} \hat{a}^{n} e^{-\eta^2/2}$$

group terms by q, number of quanta exchange:

$$= e^{-\eta^2/2} \Big\{ \sum_m (-1)^m \eta^{2m} \frac{a^{\dagger m} \hat{a}^m}{(m!)^2}$$

$$+ \sum_q \underbrace{\boxed{\sum_m (-1)^m \eta^{2m} \frac{a^{\dagger m} \hat{a}^m}{m!\,(m+q)!}}}_{f_q(\hat{a}^\dagger\hat{a})} \underbrace{\hat{a}^q (i\eta)^q}_{q \text{ dependent part}}$$

$$+ \sum_m (-1)^m \eta^{2m} a^{\dagger q} (i\eta)^q \frac{\hat{a}^{\dagger m} \hat{a}^m}{m!\,(m+q)!} \Big\}$$

for $f_q(x) = \sum_m (-1)^m \eta^{2m} \frac{x^m}{m!\,(m+q)!}$

$$= e^{-\eta^2/2} \Big\{ f_0(\hat{a}^\dagger\hat{a}) + \sum_{q>0} (i\eta)^q \left[f_q(\hat{a}^\dagger\hat{a})\,\hat{a}^q + \hat{a}^{\dagger q} f_q(\hat{a}^\dagger\hat{a}) \right] \Big\}$$

3) Full Hamiltonian:

$$\hat{H} = \omega_e |e\rangle\langle e| + \omega_t \hat{a}^\dagger\hat{a} + \frac{\Omega}{2} e^{-\eta^2/2} \Big\{ f_0(\hat{a}^\dagger\hat{a})|e\rangle\langle g| + \sum_{q>0} \cdots |e\rangle\langle g| \Big\} e^{-i\omega_L t} + hc$$

interaction picture $\hat{a} \to \hat{a}\, e^{-i\omega_t \cdot t}$ $\quad |e\rangle \to e^{-i\omega_e t} |e\rangle$

$$\hat{H} = \frac{\Omega}{2} e^{-\eta^2/2} \Big\{ f_0(\hat{a}^\dagger\hat{a})\, e^{i(\omega_e-\omega_L)t} |e\rangle\langle g| \quad \leftarrow \text{carrier}$$

$$+ \sum_q f_q(\hat{a}^\dagger\hat{a})\, e^{i(\omega_e-\omega_L-q\omega_t)\cdot t} (i\eta)^q \hat{a}^q |e\rangle\langle g| \quad \text{red sidebands}$$

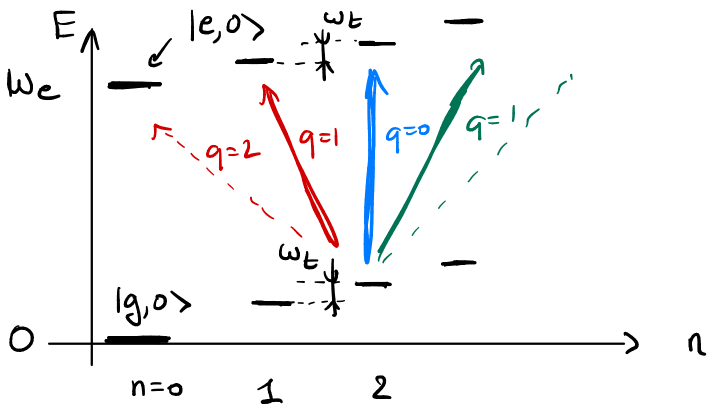
$$+ \sum_q e^{i(\omega_e-\omega_L+q\omega_t)\cdot t} (i\eta)^q \hat{a}^q |e\rangle\langle g| f_q(\hat{a}^\dagger\hat{a}) \quad \text{blue sidebands}$$

$$+ hc \Big\}$$

Each term is called a motional sideband.

- Carrier : $\omega_e - \omega_L \rightarrow$ resonant when $\omega_L = \omega_e$
- Red sidebands : $\omega_e - \omega_L - q\omega_t \rightarrow$ resonant when $\omega_L = \omega_e - q\omega_t$; $q > 0$
- Blue sidebands : $\omega_e - \omega_L + q\omega_t \rightarrow$ $= \omega_e + q\omega_t$

Summary :



Sideband q : order η^q

4) Addressing sidebands

- when $\Omega \ll \omega_t$: rotating wave approximation
  neglect terms oscillating at ω_t or higher

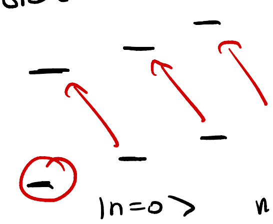
Choosing $\omega_L = \omega_e \pm q\omega_t \rightarrow$ one term!

- Lamb-Dicke regime : $\eta \ll 1$ $\quad f_q(x) = \frac{1}{q!}$, $e^{-\eta^2/2} = 1$

Carrier : $\quad H_c = \frac{\hbar\Omega}{2}\left(|g\rangle\langle e| + |e\rangle\langle g| \right)$

Red-sideband : $\quad \hat{H}_r = \frac{\hbar\Omega}{2} i\eta \left(\hat{a}\, |e\rangle\langle g| - \hat{a}^\dagger\, |g\rangle\langle e| \right)$ $\quad$ Jaynes-Cummings Hamiltonian

Remark : there exists a "dark" state



$|n=0\rangle$ not coupled to light

Blue sideband : $\quad \hat{H}_b =$

Anti-Jaynes-Cummings Hamiltonian
no dark state.

II - Sideband cooling

1) Cooling cycle

Lindblad equation : $\quad \dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}\,|e\rangle\langle e| \right) + \sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^\dagger$

with : $\quad \hat{L}_{\vec{n}} = e^{ik\vec{n}\cdot\hat{\vec{r}}} |g\rangle\langle e|$

in 1D: $\hat{L}_\pm = e^{\pm i k \hat{z}} |g\rangle\langle e|$

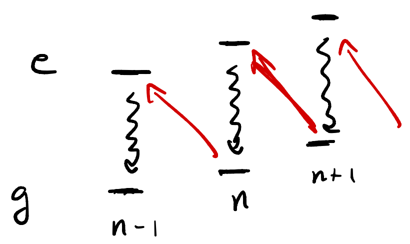
$$= e^{\pm i k z_0 (\hat{a} + \hat{a}^+)} |g\rangle\langle e|$$

$$= \left(\hat{\mathbb{1}} \pm i\eta(\hat{a}+\hat{a}^+) + \dots\right) |g\rangle\langle e|$$

Same as free space:

$$\sum_{\vec{n}} \Gamma_{\vec{n}} \hat{L}_{\vec{n}} \hat{\rho} \hat{L}_{\vec{n}}^+ \sim \sum_{\vec{n}} \Gamma_{\vec{n}} |g\rangle\langle e| \hat{\rho} |e\rangle\langle g| + O(\eta^2)$$

<u>in the Lamb-Dicke regime $\eta \ll 1$</u>: Spontaneous emission does not change n



Driving the red sideband + sp. emission

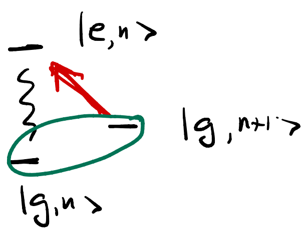
$|n\rangle \to |n-1\rangle$

cooling! Down to $|n=0\rangle$

(in practice : limited by $O(\eta^2)$)

2) Quantitative description

Method of adiabatic elimination



3 levels problem

for $\Omega_r \ll \Gamma$: $\rho_{ee} = 0$

→ effective 2 level theory:

$$\dot{\hat{\rho}} = -i[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}\left(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e|\right) + \Gamma \underbrace{\langle e|\hat{\rho}|e\rangle}_{\text{operator on } \mathcal{H}_{com}} |g\rangle\langle g|$$

- $\langle e|\dot{\hat{\rho}}|e\rangle = \dot{\hat{\rho}}_{ee}$

$$= -i \langle e| \left[\frac{i\Omega_r\eta}{2}\left(\hat{a}|e\rangle\langle g| - \hat{a}^+|g\rangle\langle e|\right), \hat{\rho} \right] |e\rangle - \Gamma\hat{\rho}_{ee}$$

$$= \frac{\Omega_r\eta}{2}\left(\hat{a}\underbrace{\langle g|\hat{\rho}|e\rangle}_{\hat{\rho}_{ge}} + \underbrace{\langle e|\hat{\rho}|g\rangle}_{\hat{\rho}_{eg}}\cdot\hat{a}^+\right) - \Gamma\hat{\rho}_{ee}$$

- $\langle g|\dot{\hat{\rho}}|g\rangle = \dot{\hat{\rho}}_{gg}$

$$= -\frac{\Omega_r\eta}{2}\left(\hat{\rho}_{ge}\hat{a} + \hat{a}^+\hat{\rho}_{eg}\right) + \Gamma\hat{\rho}_{ee}$$

- $$\dot{\hat{\rho}}_{eg} = \frac{\Omega_r\eta}{2}\left(\hat{a}\hat{\rho}_{gg} + \hat{\rho}_{ee}\hat{a}^+\right) - \frac{\Gamma}{2}\hat{\rho}_{eg}$$

Internal degrees of freedom are fast!

Steady state is reached : $\hat{\rho}_{eg} = \frac{i\Omega_r \eta}{\Gamma} \hat{a} \hat{\rho}_{gg} \Leftarrow \hat{\rho}_{ee} \sim 0$ (to order $\frac{\Omega_r^2}{\Gamma^2}$)

$$\dot{\hat{\rho}}_{gg} = -\frac{\Omega_r^2 \eta^2}{2\Gamma} \left(\hat{a}^\dagger \hat{a} \hat{\rho}_{gg} + \hat{\rho}_{gg} \hat{a}^\dagger \hat{a} \right) + \frac{\Omega_r^2 \eta^2}{\Gamma} \hat{a} \hat{\rho}_{gg} \hat{a}^\dagger$$

Effective Lindblad equation for $|g\rangle$:

$$\boxed{\dot{\hat{\rho}} = -\frac{\Gamma_1}{2} \left(\hat{a}^\dagger \hat{a} \hat{\rho} + \hat{\rho} \hat{a}^\dagger \hat{a} \right) + \Gamma_1 \hat{a} \hat{\rho} \hat{a}^\dagger}$$

$$\hat{L}_{eff} = \sqrt{\Gamma_1}\, \hat{a} \qquad \Gamma_1 = \frac{\Omega_r^2 \eta^2}{\Gamma} \ll \Gamma, \Omega_r$$