Introduction to Particle Accelerators, Tutorial 11

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Exercise 1. Consider a storage ring composed of $N = 24$ identical FODO cells with the parameters given in Tab. 1.

Parameter		**Value**	
Half-cell length	l_d	6	m
Quadrupole length	l_q	0.705	m
Quadrupole strength	k	0.3659	m^{-2}

Table 1: FODO parameters.

(a) Is the transverse particle motion in such a structure stable? Justify your answer.
Yes. Stability requires $f > L_{\text{cell}}/4$, where
$L_{\text{cell}} = 2l_d = 12\,\text{m}$, and $f = \frac{1}{kl_q} = 3.88\,\text{m}$.

(b) Calculate the betatron tunes of the machine for the given parameters. Assume equal quadrupole strengths in both planes.

It is possible to calculate the phase advance per FODO cell by using: $\sin(\frac{\mu}{2}) = \frac{L_{cell}}{4f}$. Therefore the tune will be: $Q_{x,y} = N \cdot \frac{\mu}{2\pi} = 6.75$ *(same in both planes, since structure is symmetric)*

(c) Consider the thin-lens approximation and calculate the values of the β-function at the center of the focusing and defocusing quadrupoles.

The β_x-function is maximum at the focusing quads and minimum at the defocusing quads, and vice-versa for β_y. The max. and min. values are
$\beta_{\text{max}} = 21.7\,\text{m}$, and $\beta_{\text{min}} = 2.8\,\text{m}$.

Exercise 2. The LHC lattice in the arcs consists of a symmetric FODO structure with an elliptical pipe cross-section with physical aperture dimensions of $r_x = 22\,\mathrm{mm}$ and $r_y = 18\,\mathrm{mm}$. Table 2 provides typical values of the main optics parameters in the arcs at injection energy. All values are given in units of m. The total energy at injection is 450 GeV and the proton rest mass is $938.272\,\mathrm{MeV}/c^2$.

Max. $\boldsymbol{\beta_x}$	Min. $\boldsymbol{\beta_x}$	Max. $\boldsymbol{\beta_y}$	Min. $\boldsymbol{\beta_y}$	Max. $\mathbf{D_x}$	Min. $\mathbf{D_x}$
172.3	14.8	176.5	13.2	2.4	-0.1

Table 2: Main optics parameters at injection energy. All values in units of m.

(a) Where are the minima and maxima of β_x, β_y, and D_x?

(b) Calculate the horizontal and vertical acceptance at injection energy considering monochromatic beam.

For monochromatic beam there is no dispersion ($\delta p/p = 0$).

The horizontal acceptance is:

$$\epsilon_{Ax} = \frac{r_x^2}{\beta_x^{max}} = 2.81\,\mu\mathrm{m}$$

and the vertical acceptance:

$$\epsilon_{Ay} = \frac{r_y^2}{\beta_y^{max}} = 1.84\,\mu\mathrm{m}$$

(c) Compute the maximum number of rms beam sizes that fit in the vacuum chamber for both transverse planes at injection energy. Consider an energy spread of $\sigma_\delta = 1.129 \times 10^{-4}$. The normalized transverse beam emittance is $\epsilon_n = 2.5\,\mu\mathrm{m}$ rad in both planes.

This time we need to include the effect of dispersion in the beam size by using $\sigma = \sqrt{\epsilon_{geo}\beta_{x,y} + D_{x,y}^2\sigma_\delta^2}$. First we need calculate the geometric emittance:

$$\epsilon_{geo} = \frac{\epsilon_n}{\beta\gamma} = 5.211 \times 10^{-9}\,\mathrm{m}$$

The horizontal maximum beam size is:

$$\sigma_{x,max} = \sqrt{\epsilon_{geo} \cdot \beta_{x,max} + D_{x,max}^2 \cdot \sigma_\delta^2} = 9.8554 \times 10^{-4}\,\mathrm{m}$$

The maximum vertical beam size:

$$\sigma_{y,max} = \sqrt{\epsilon_{geo} \cdot \beta_{y,max}} = 9.590 \times 10^{-4}\,\mathrm{m}.$$

Therefore the number of rms beam sizes that fit in the in the vacuum chamber:

$$N_{\sigma_x} = \frac{r_x}{\sigma_{x,max}} = 22.3$$

$$N_{\sigma_y} = \frac{r_y}{\sigma_{y,max}} = 18.8.$$

Exercise 3. Consider a 5 TeV electron storage ring built around the Earth's equator (diameter $D = 11000\,\text{km}$). Assume that the storage ring is completely filled with bending magnets.

(a) Calculate the momentum, Lorentz factor γ, the electron velocity β, and the revolution period T_0 of the beam.
$p \approx 5\,\text{TeV/c}$
$\gamma = 9.785 \times 10^6$
$\beta = 1 - 1.044 \times 10^{-14} \approx 1$
$T_0 = 0.115\,\text{s}$

(b) Estimate the magnetic field strength in the bending magnets.
$\rho = D/2 = 5.5 \times 10^6\,\text{m}$ and $B = 3\,\text{mT}$.

(c) Calculate the energy loss per turn U_0 due to synchrotron radiation.

$U_0 = 88.5 \times 10^{-5} \frac{E^4[GeV^4]}{\rho[m]} = 10.1\,\text{TeV}$

(d) Compute the transverse damping time in seconds and in number of turns.
$\tau_{x,y} = \frac{2ET_0}{U_0} = 0.114\,\text{s} < 1\,\text{turn}.$

A more moderate proposal: use only the Earth's magnetic field ($B = 3 \times 10^{-5}\,\text{T}$, with direction parallel to the Earth's rotational axis) instead of bending magnets and let the electrons circulate in the same vacuum pipe as above.

(e) Determine the new energy of the electron beam.
$E_{\text{tot}} = 49.47\,\text{GeV}$

(f) Evaluate again the energy loss per turn due to synchrotron radiation.
$U_0 = 95.5\,\text{keV}$

(g) Calculate the critical energy of the synchrotron radiation photons.
$\epsilon_{\text{crit}} \approx 48.8\,\text{eV}$

(h) Assuming an energy acceptance of the machine of $\Delta E/E = 1\,\%$, how many full revolutions will the beam perform if no RF acceleration is provided?

$n = 0.01E/U_0 \approx 5128\,\text{turns}.$

Exercise 4. Protons are accelerated in a two-gap cyclotron whose magnet has a diameter of $D = 2\,\text{m}$ and a magnetic dipole field of $B = 1\,\text{T}$.

(a) What is the maximum kinetic energy that can be achieved with this accelerator?
$E_{\text{kin}} = 46.7\,\text{MeV}$

(b) Assume that the maximum energy is achieved after 100 turns. What is the energy gain at every passage through the cavity gap?

$\Delta E_{\text{gap}} = E_{kin}/200 = 234\,\text{keV}$ since the protons passes through the RF 2 times per turn.

(c) A proton with an orbit of $R = 0.5\,\mathrm{m}$ is synchronous with the RF system. How many times did the proton travel across the RF gap?

$p = B\rho/3.3356 = 1\mathrm{T} \cdot 0.5\mathrm{m}/3.3356 = 0.15\,\mathrm{GeV}.$

$E_{tot} = \sqrt{p^2 + m^2}$=0.9491 GeV,
$E_{kin} = E_{tot} - m^2$=0.0119 GeV
$n_{\mathrm{gap}} = E_{kin}/\Delta E_{\mathrm{gap}} \approx 51.$

(d) What is the oscillation period of the RF system?
$T_0 = T_{\mathrm{RF}} \approx 6.64 \times 10^{-8}\,\mathrm{s}.$

Exercise 5. An electron beam of 5 GeV is sent through a long 2 T bending magnet, and through an undulator with a period of 15 mm. At the given magnetic gap (separation of undulator jaws), the undulator provides a magnetic field of 0.5 T on axis.

(a) What is the critical photon wavelength of the synchrotron radiation emitted in the bending magnet?

Using $B\rho\,[\mathrm{T\ m}] = 3.3356\,p\,[\mathrm{GeV/c}]$, we first compute $\rho = 8.339\,\mathrm{m}$ (note that since $E \gg m_e c^2$, $p \approx E$). We can then use the formula for the critical frequency $\omega_{\mathrm{crit}} = \frac{3}{2} c \frac{\gamma^3}{\rho}$, and combine it with $\lambda = \frac{2\pi c}{\omega}$ to calculate the critical wavelength

$$\lambda_{\mathrm{crit}} = \frac{4\pi\rho}{3\gamma^3} = 37.3\,\mathrm{pm},$$

with $\gamma = E_{\mathrm{tot}}/(m_e c^2) = 9785$.

(b) What is the main frequency of the synchrotron radiation produced in the undulator in the forward direction?

From the lectures, we know for the main harmonic ($n = 1$)

$$\lambda = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2} + \gamma^2\theta^2\right).$$

Since we are interested in the forward direction only, $\theta = 0$. The strength parameter K is also provided in the lectures as

$$K = \frac{eB\lambda_u}{2\pi m_e c} \approx 0.0934\,\lambda_u\,[\mathrm{mm}]\,B\,[\mathrm{T}].$$

With the given B-field and undulator period, we obtain $K \approx 0.7$ and $\lambda = 97.5\,\mathrm{pm}$.

(c) Describe qualitatively what happens to the photon energy (of the main harmonic) when we increase the undulator gap (*hint: how does the magnetic field of the undulator change?*).

When increasing the undulator gap, the on-axis B-field in the undulator is reduced, and hence K is reduced as well. This leads to a decrease in λ, and hence to an increase in the (main) photon energy.

(d) You would like to produce a broader spectrum of synchrotron radiation and hence want to turn your undulator ($K \leq 1$) into a wiggler ($K > 1$). How could you potentially achieve that?

It depends. We can, for example, decrease the magnetic gap to increase the B-field in the device. This will increase the strength parameter K. It is possible that decreasing the magnetic gap will not be enough to reach $K > 1$, however. In that case, we would have to develop a new device (or alter the existing one substantially), by increasing the dipole strength of the permanent magnets, and / or increasing the magnetic period λ_u.