Case Studies

[three concrete examples of accelerators]

Laboratory for Particle Accelerator Physics, EPFL

The Goal for this lesson

apply the studied concepts to realistic applications, determine the key parameters for three accelerator configurations

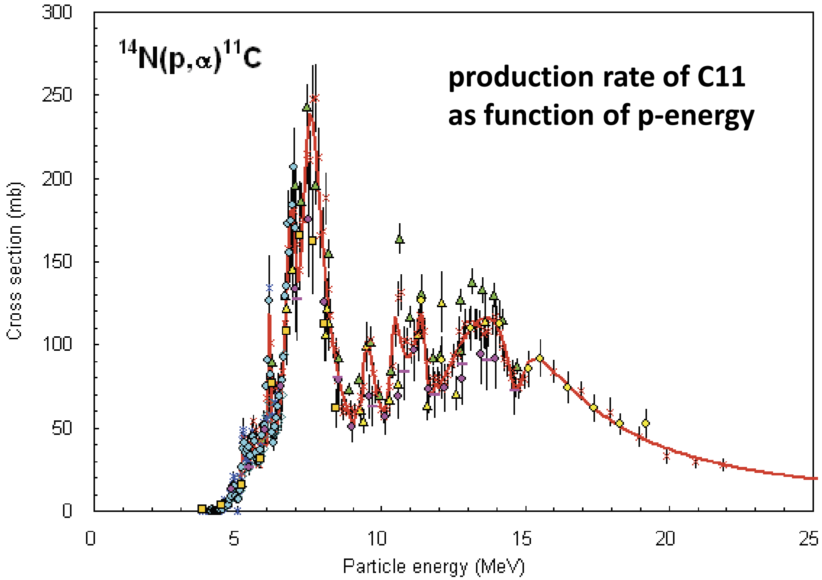
1. design of a compact cyclotron for medical isotope production [**10MeV**]
2. parameters of a high energy proton accelerator [**500GeV**]
3. use the same synchrotron for electrons **[20GeV]**

→ recapitulation of several accelerator concepts

Case Study : 10MeV Cyclotron

- motivation
- Step 1: define bending magnet
- Step 2: focusing properties
- Step 3: realise acceleration to full energy

Isotope Production for medical Purposes



Carbon-11 (C-11) radiotracers are widely used for the early diagnosis of cancer, monitoring therapeutic response to cancer treatment, and pharmacokinetic investigations of anticancer drugs.

The short half-life of carbon-11 (20.38 minutes) creates special challenges for the synthesis of C-11 labeled tracers; these include the challenges of synthesizing C-11 target compounds with high radiochemical yield …

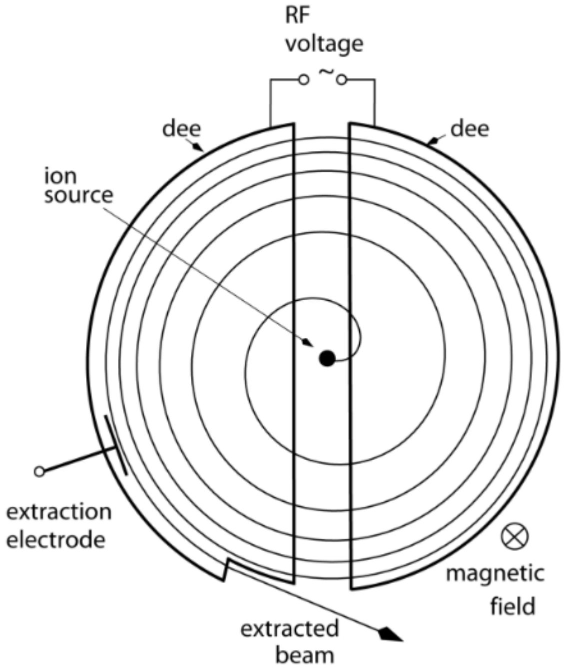
[Z. Tu, R.H. Mach, Washington University School of Medicine]

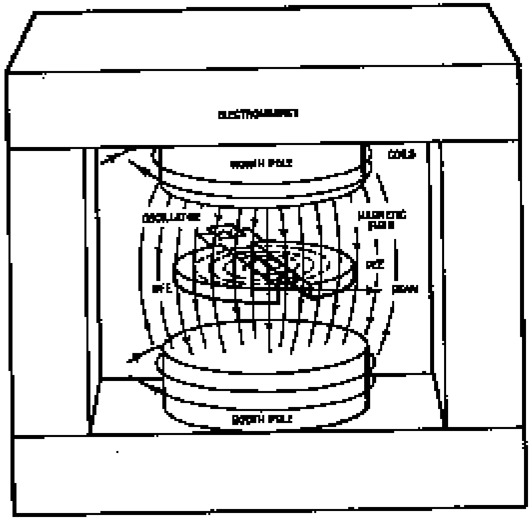
→ we need **a super compact and cost efficient accelerator** delivering ≈ 10MeV protons, suited for installation in a hospital

The Classical Cyclotron

- two capacitive electrodes „Dees“, two gaps per turn
- internal ion source
- homogenous B field, radial gradient possible
- **constant revolution frequency for low E_k**

our plan: accelerate fast and survive with a slight non-synchronism



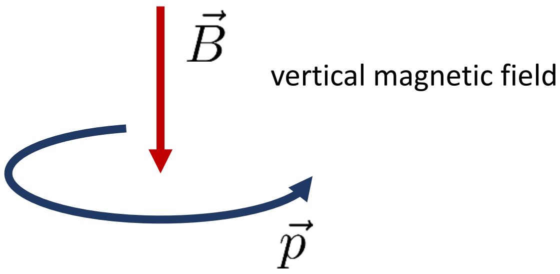


Lorentz force in magnetic field

$$\vec{F} = e\,\dot{\vec{r}} \times \vec{B} + \cancel{e\vec{E}} = m\,\omega^2\vec{r}$$

force perpendicular to velocity $\dot{\vec{r}}$ and field $\vec{B}$

centrifugal force

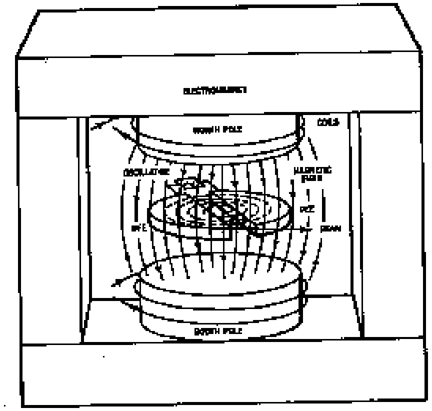


circular path

Solution is circular path:

$$\vec{r}(t) = \rho \begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix} \quad \text{with:} \quad \rho = \frac{p}{eB_z} \qquad \omega = \frac{eB_z}{m}$$

note: $m = \gamma m_0$



Relativistic Factors

relativistic energy-momentum relation:

$$\boxed{\begin{aligned} E_{\mathrm{tot}} &= \sqrt{c^2p^2 + m_0^2c^4} \\ E_{\mathrm{tot}} &= E_k + m_0c^2 \\ &= \gamma\, m_0c^2 \end{aligned}}$$

↑

proton rest energy:
938 MeV

convenient relativistic factors:

$$\boxed{\begin{aligned} \gamma &= 1 + \frac{E_k}{m_0c^2} \\ \beta &= \sqrt{1 - \frac{1}{\gamma^2}} \end{aligned}}$$

momentum: $p = mv$

$$\begin{aligned} cp &= \beta\gamma\, m_0c^2 \\ &= \beta E_{\mathrm{tot}} \end{aligned}$$

Step 1: Choice of Field for Energy ≈ 10MeV

C. Oliver, CIEMAT, Spain et al.
superconducting cyclotron, B=4 Tesla

magnetic rigidity

$$B\rho\,[\mathrm{T}\cdot\mathrm{m}] = 3.3356 \cdot p\,[\mathrm{GeV/c}]$$

not negligible in this case, β<<1

$$B\rho\,[\mathrm{T}\cdot\mathrm{m}] = 3.3356 \times \beta \times E_{\mathrm{tot}}\,[\mathrm{GeV}]$$

$$\left(\approx 4.426\sqrt{E_k/m_o c^2} \text{ for } \beta \ll 1\right)$$

kinetic energy E_k	10 MeV
velocity β	0,145
superconducting magnet	4.0 T
extraction radius @ 4T	11,4 cm

The high field of 4T results in compact dimensions.

Step 2: Focusing in Classical Cyclotron

Even with a simple magnet we can introduce a radial gradient to achieve focusing.

the **field index** describes the (normalized) radial slope of the bending field:

$$k = \frac{R}{B}\frac{dB}{dR}$$

we want focusing in both planes:

$$\ddot{x} + \omega_c^2(1+k)x = 0$$

$$\ddot{y} - \omega_c^2\, ky = 0$$

tunes in a cyclotron, both should real numbers:

$$Q_r = \sqrt{1+k}$$

$$Q_y = \sqrt{-k}$$

thus -1 < k < 0 (negative slope of field) to keep beam focused!
but: not isochronous!

Overview on cyclotron type accelerators

Classical Cyclotron*
[B(θ) = const]

*** our choice**

Thomas cyclotron
[Azimuthally Varying Field,
e.g. $B(\theta) \propto b+\cos(3\theta)$, one pol]

complexity

AVF concept – harmonic pole shaping,
electron model, Richardson et al (1950),
courtesy of Lawrence Berkeley National Laboratory

Separated Sector Cyclotron
[separated magnets, resonators]

Synchro-Cyclotron
[varying RF frequency]

Fixed Field Alternating Gradient Accelerator FFAG
[varying RF, strong focusing]

high intensity

high energy

compact machine

Obtaining a minimal focusing in vertical plane

parametrization of B:

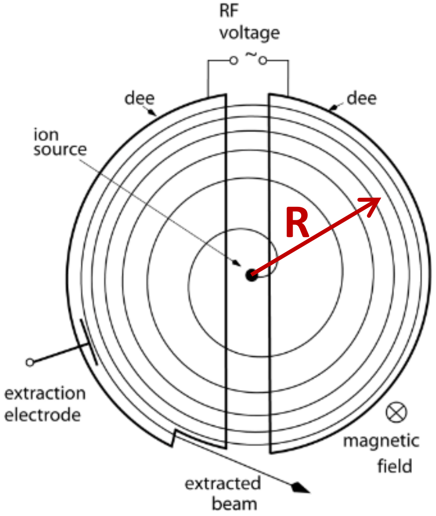
$$B(R) = B_0 \left(\frac{R}{R_0}\right)^k$$

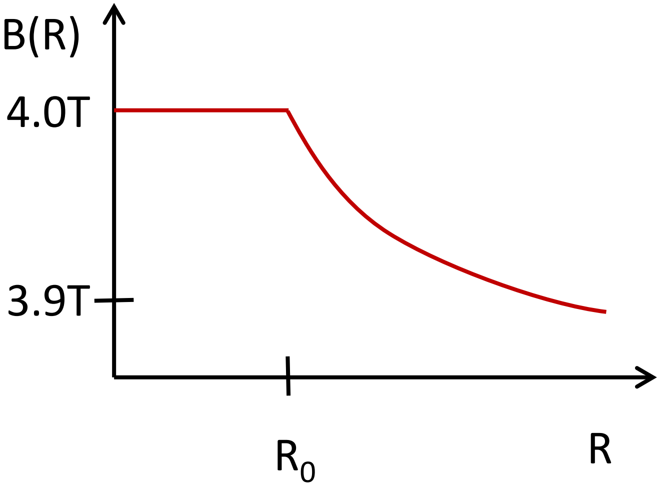
$$k = \frac{R}{B}\frac{dB}{dR}$$

→ a small for value for k is sufficient (compromise between focusing strength and phase slip)

→ chose k = -0.01

→ Q_y = 0.1 → one oscillation per 10 turns





Step 3: Acceleration Process

very simple simulation process, for example using Excel:

set of equations to iterate through the acceleration process:

$$\Delta E_k = V_g \times \sin\phi$$

$$\Delta\phi = 2\pi \times \frac{B(R(E_k)) - B_{\mathrm{iso}}}{B_{\mathrm{iso}}}$$

B_{iso} = field theoretically required to obtain a fixed revolution frequency f_{rev}

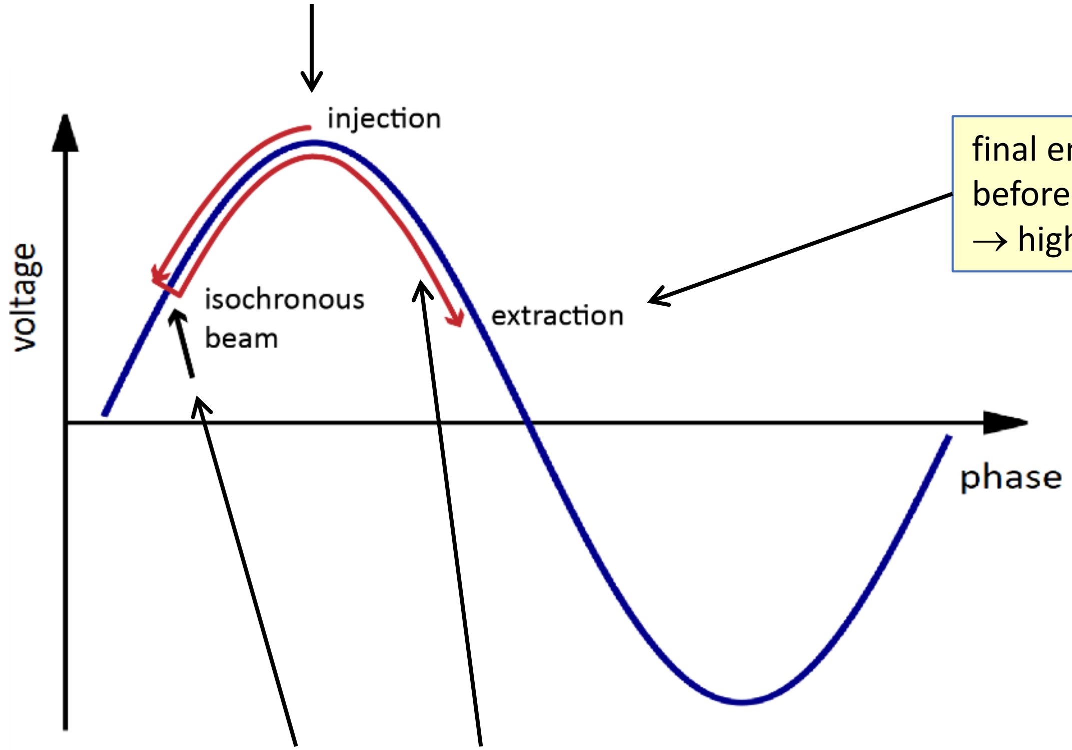
$$B_{\mathrm{iso}} = \frac{2\pi\gamma m_0}{e} f_{\mathrm{rev}}$$

energy and phase are computed from turn to turn

→ the aim is to see whether the desired energy can be reached without losing completely the synchronism with the RF wave

Qualitative Explanation of Acceleration in a non-isochronous Cyclotron

field to high, RF frequency to low for the beam, beam moves earlier



final energy must be reached before phase equals 180deg
→ high enough voltage needed

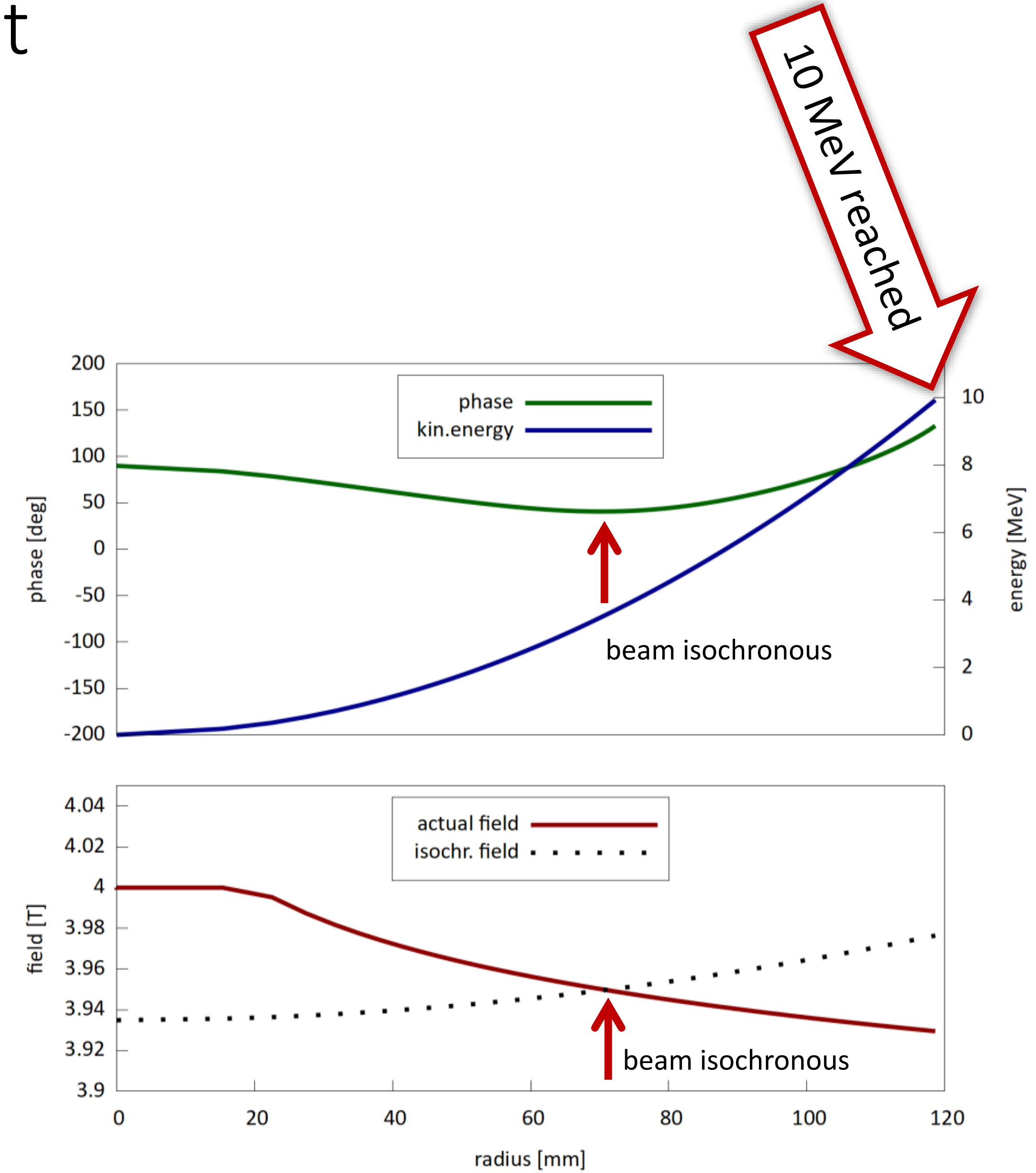
at higher energy & larger radius (B↓) circulation time exactly matches RF freq.

at even larger radius RF frequency is too high, beam moves later in phase

sketch by F.Chautard (GANIL) and W.Kleeven (IBA)

Simulation Result

result:
with 2x46kV voltage gain per turn
an energy of 10MeV is reached
Nr of turns: 65

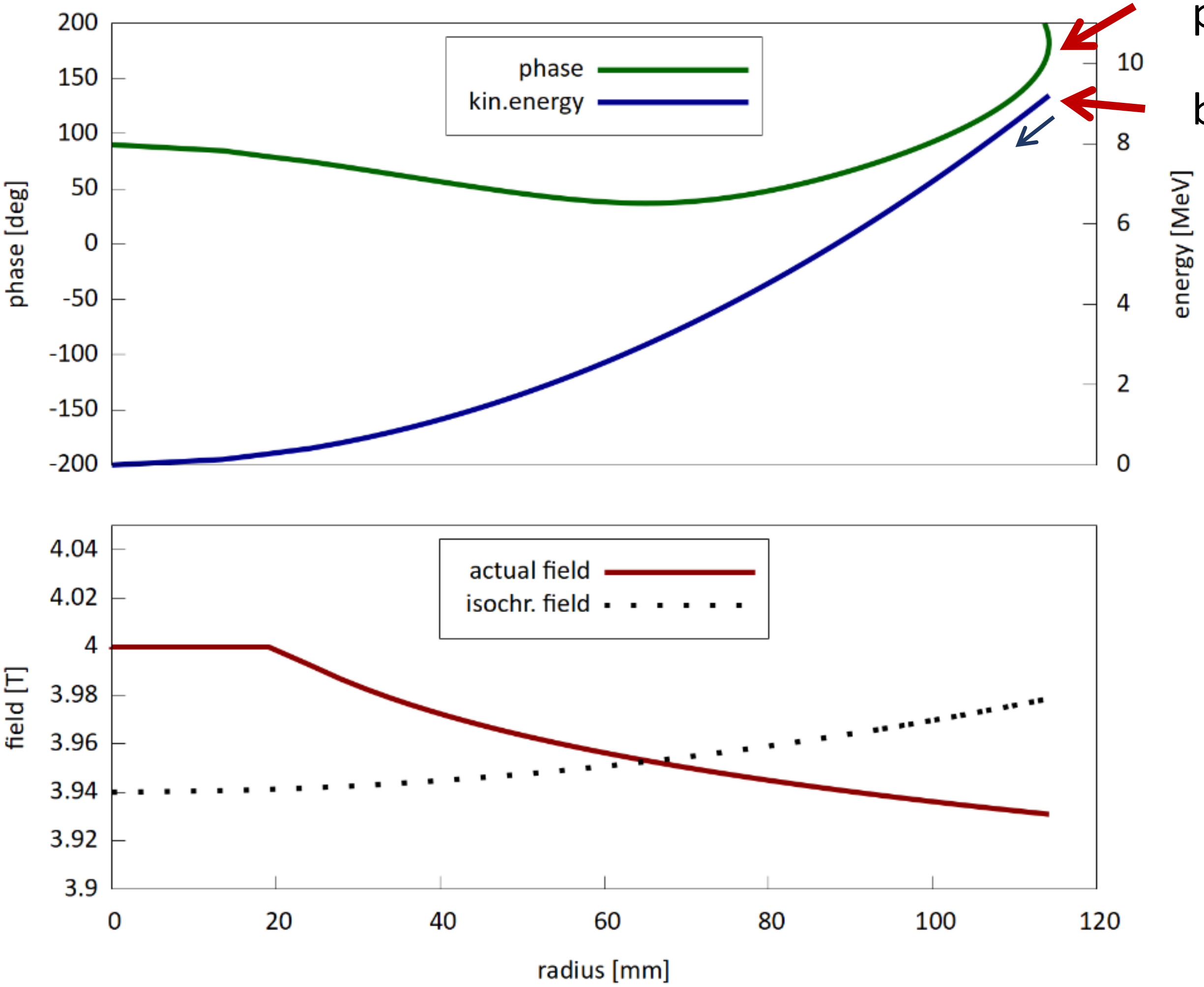


10 MeV reached

beam isochronous

beam isochronous

Illustration of insufficient voltage



phase of 180deg reached

beam decelerated

illustration:
with 70kV voltage the phase turns over and beam is decelerated again
E_{max} falls below 10MeV

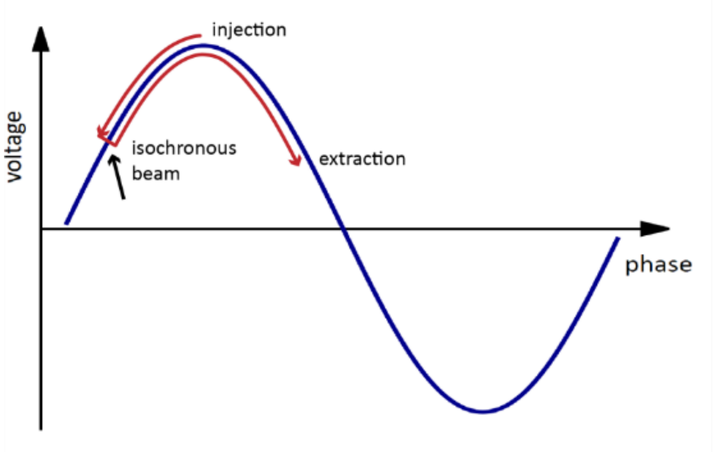
Summary on Cyclotron Example

using a high field of a superconducting magnet a compact and simple 10MeV cyclotron for isotope production can be realised

in this classical cyclotron the beam stays not in phase with the RF, thus a high gap voltage >45kV is needed to perform the acceleration quickly

we needed:

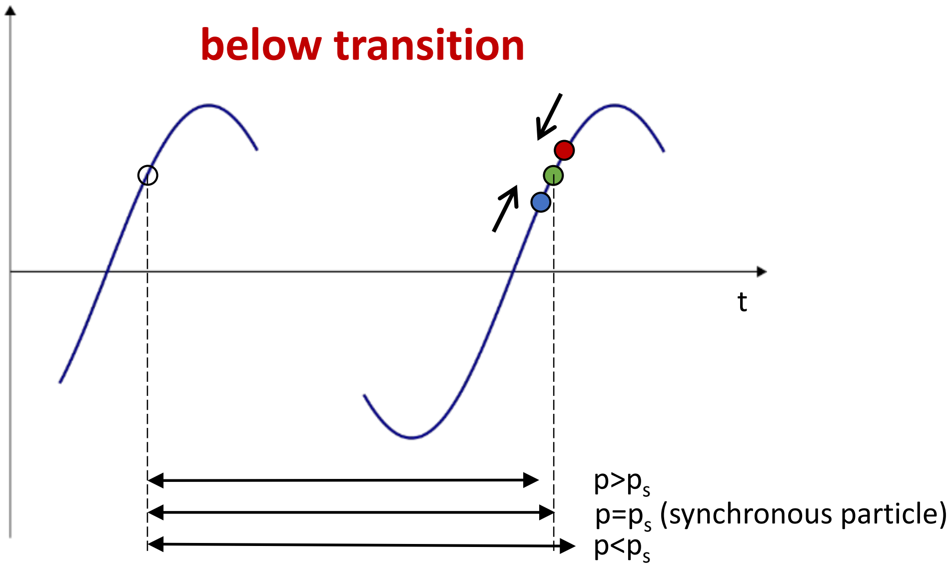
- **magnetic rigidity, cyclotron frequency**
- **knowledge on focusing via radial field index**
- **a bit of longitudinal dynamics and ..**
- **numerical integration of energy, phase coord.**



Recap Synchrotron:
Stable phases above and below transition ?

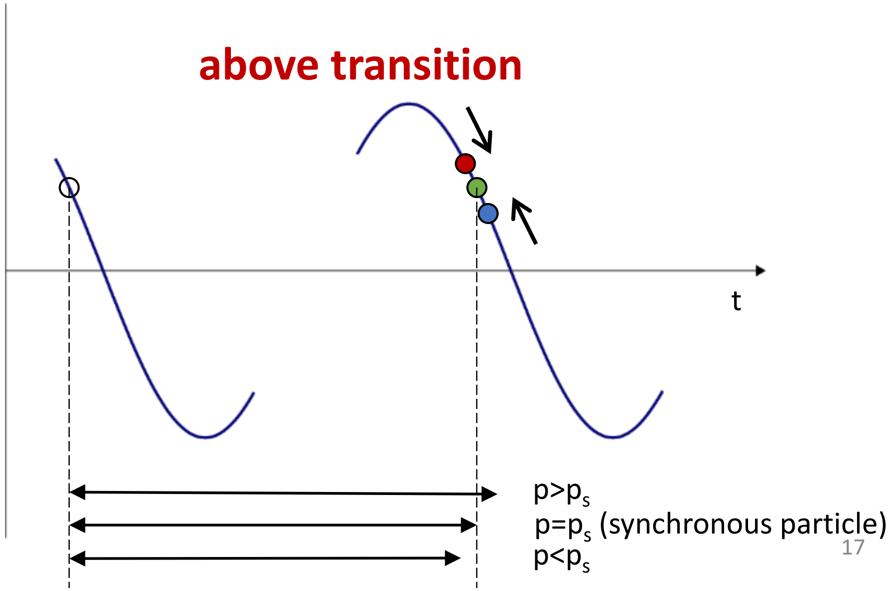
$E<E_{tr}$: with increasing energy circulation time is reduced (dominated by velocity)

$\eta_c < 0$



$E>E_{tr}$: with increasing energy circulation time is increased (dominated by path length)

$\eta_c > 0$



Synchrotron: Transition Energy

dependence of circulation time on momentum

competing effects: **path length** ↔ **speed of particles**

$$\boxed{\frac{\Delta\tau}{\tau} = \underbrace{\left(\frac{1}{\gamma_{\mathrm{tr}}^2} - \frac{1}{\gamma^2}\right)}_{\text{Slip factor: } \eta_c} \frac{\Delta p}{p}}$$

$$\frac{1}{\gamma_{\mathrm{tr}}^2} = \overset{\substack{\text{momentum compaction factor} \\ \downarrow}}{\alpha_c} = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} ds$$

$$E_{\mathrm{tr}} = \gamma_{\mathrm{tr}} m_0 c^2$$

the **transition energy** is a parameter of the magnet lattice

---

compare **cyclotron** example:

$$\frac{\Delta\tau}{\tau} = \underbrace{\left(\frac{1}{1+k} - \frac{1}{\gamma^2}\right)}_{\text{= 0 for isochronous cyclotron}} \frac{\Delta p}{p} \qquad k = \frac{R}{B}\frac{dB}{dR} \quad \text{field index}$$

Stationary Bucket and Separatrix

relation with slip factor:

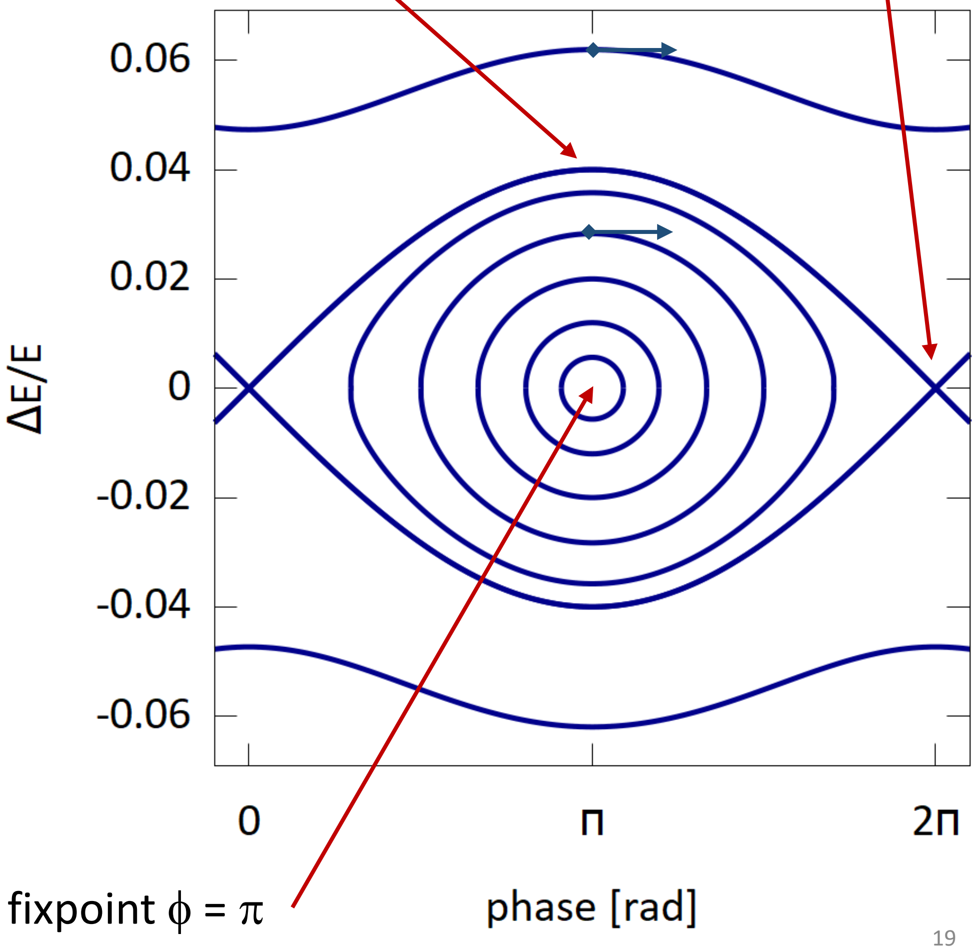
$$\frac{\Delta\tau}{\tau} = \eta_c \frac{\Delta p}{p}$$

for example above transition:

$\eta_c > 0 \rightarrow \Delta E > 0,\ \Delta\phi > 0$

maximum energy acceptance on separatrix

unstable fixpoint defines H_{sep}, separatrix



Stable fixpoint $\phi = \pi$

see also Wiedemann sec. 9.3.2

Next: Case Study 500GeV Synchrotron

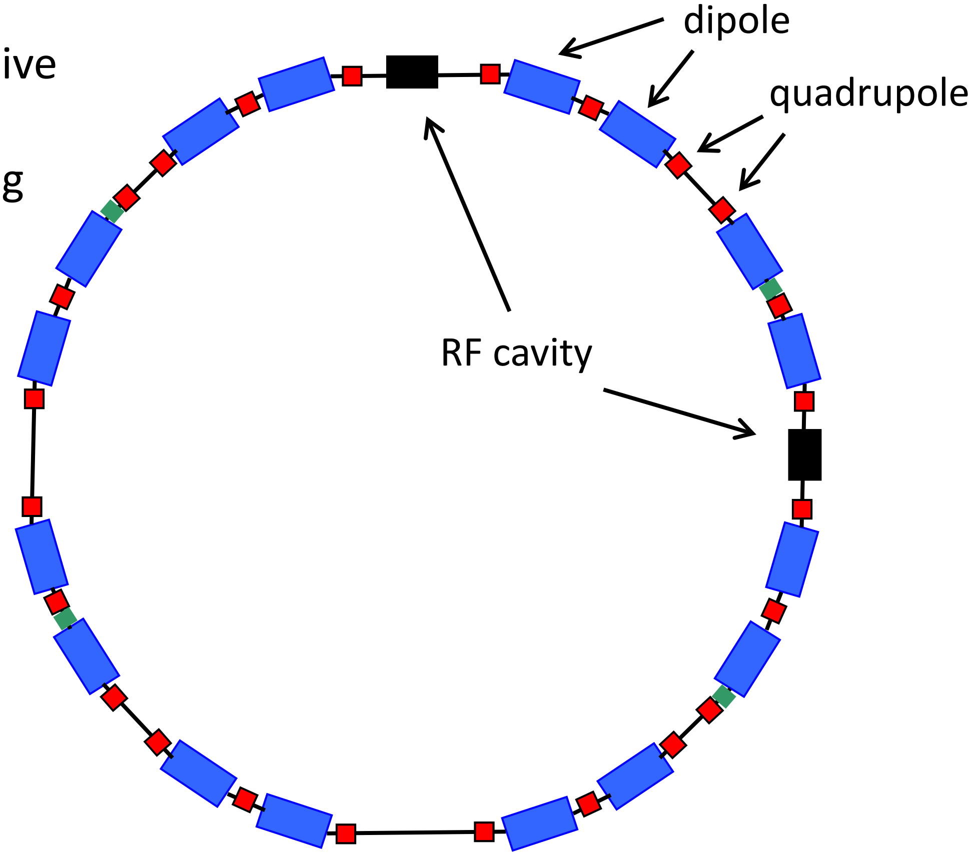
- bending strength, injection/extraction energy
- focusing and beam size
- main parameters, chromaticity and its correction

Common Accelerator Types: Synchrotron

Synchrotron:

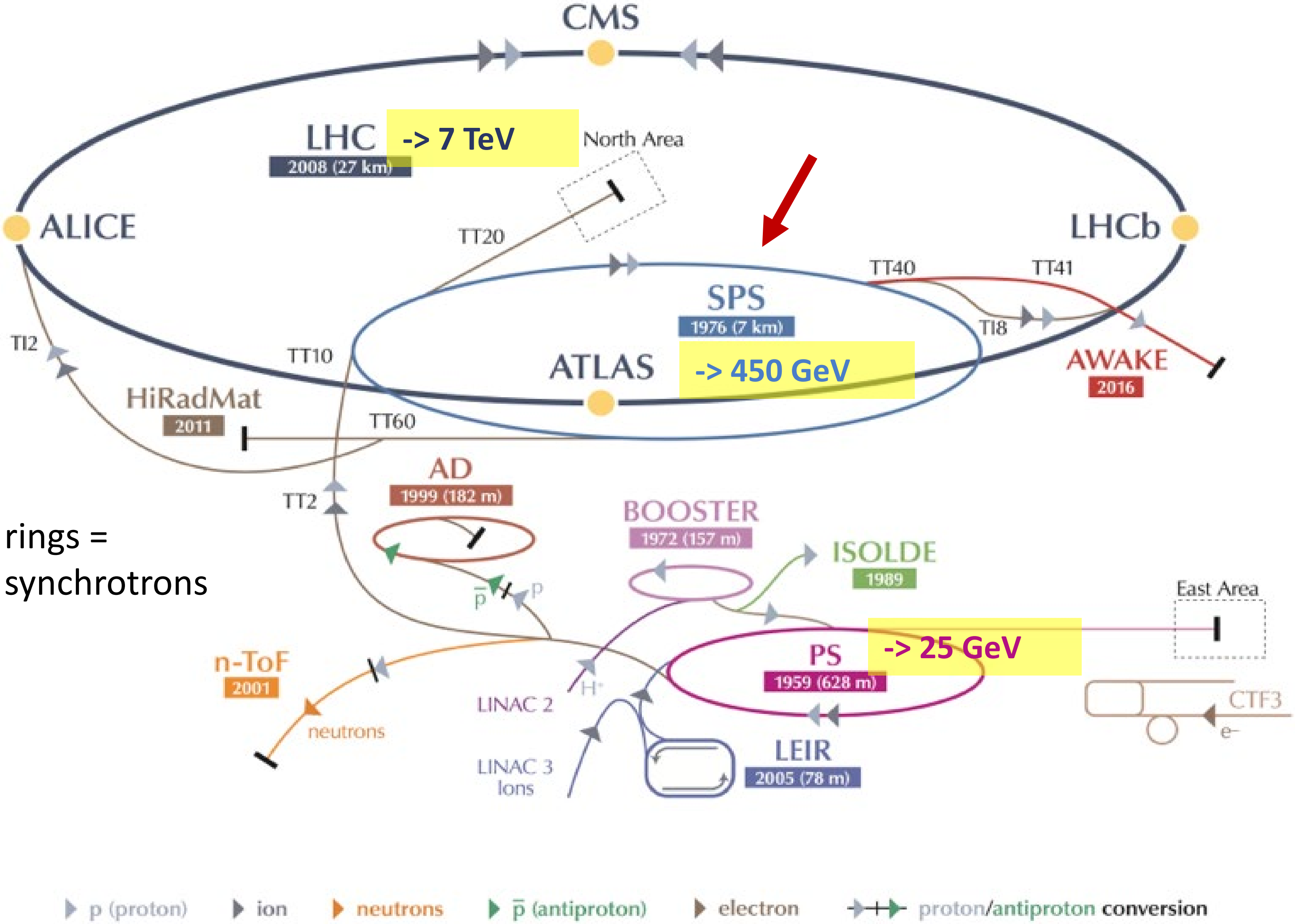
- multiple use of cavities
- high energies possible – limited by B-field (p, ions) or SR power (e)
- for a range of parameters cost effective

→ this is the best choice for accelerating protons to 500GeV



CERN Accelerator Zoo

SPS = Super Proton Synchrotron, 25 → 450GeV



-> 7 TeV

-> 450 GeV

-> 25 GeV

rings = synchrotrons

A high energy proton synchrotron a la SPS

a proton synchrotron for 500GeV

→ ring size and field strength
→ FODO lattice choices
→ other machine parameters

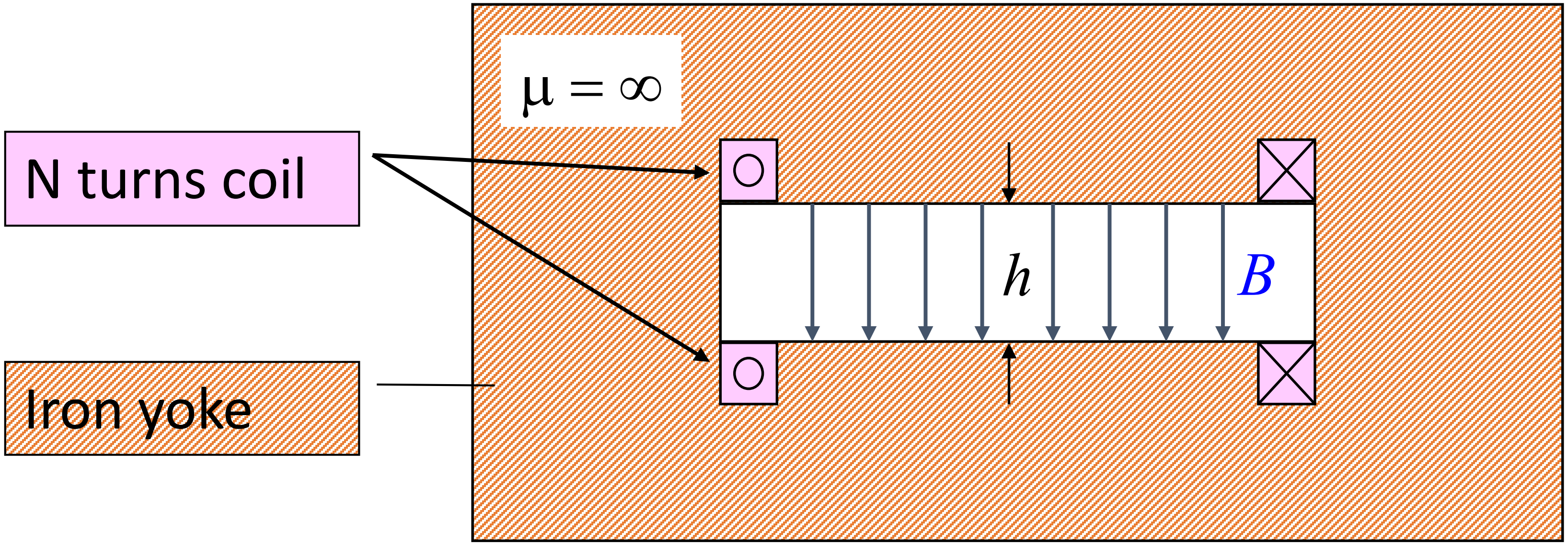
[Super Proton Synchrotron, CERN]

main dipoles of the SPS at CERN: 2.0 T × 6.3 m

[A.Milanese, CERN]

Bending magnets (dipoles)

Iron dominated magnets (B < 2 Tesla)



[cross section perpendicular to beam path]

field strength in gap: $$B = \mu_0 \frac{N \cdot I}{h}$$

I magnet current
N number of turns
h gap height

Step 1: A Magnet for a 500GeV Ring

length = 12m

room temp. magnet: B_{max} = 2T

→ bend radius ρ_{min} = 835.5m

→ bending angle $\theta = l/\rho$ = 14.4mrad

magnetic rigidity:

$$B\rho\,[\mathrm{T}\cdot\mathrm{m}] = 3.3356 \cdot p\,[\mathrm{GeV/c}]$$

magnets needed for ring: $N = 2\pi/\theta$ = 437.4
→ rounding up: N = 440 Magnets

magnet specification:
B = 1.988 Tesla, l = 12.0 m, N = 440

→ ρ = 840.4m

Step 2: Energy Range

the synchrotron ramps from an injection energy E_{min} to E_{max} = 500GeV

due to field errors etc. **the factor E_{max}/E_{min} cannot be arbitrarily large**

we coose E_{min} = 25 GeV, which is reasonably possible

the normalized emittance
is conserved:

$$\varepsilon_n = \beta\gamma\varepsilon \approx \gamma\varepsilon$$

$$\varepsilon_n = \mathrm{const}$$

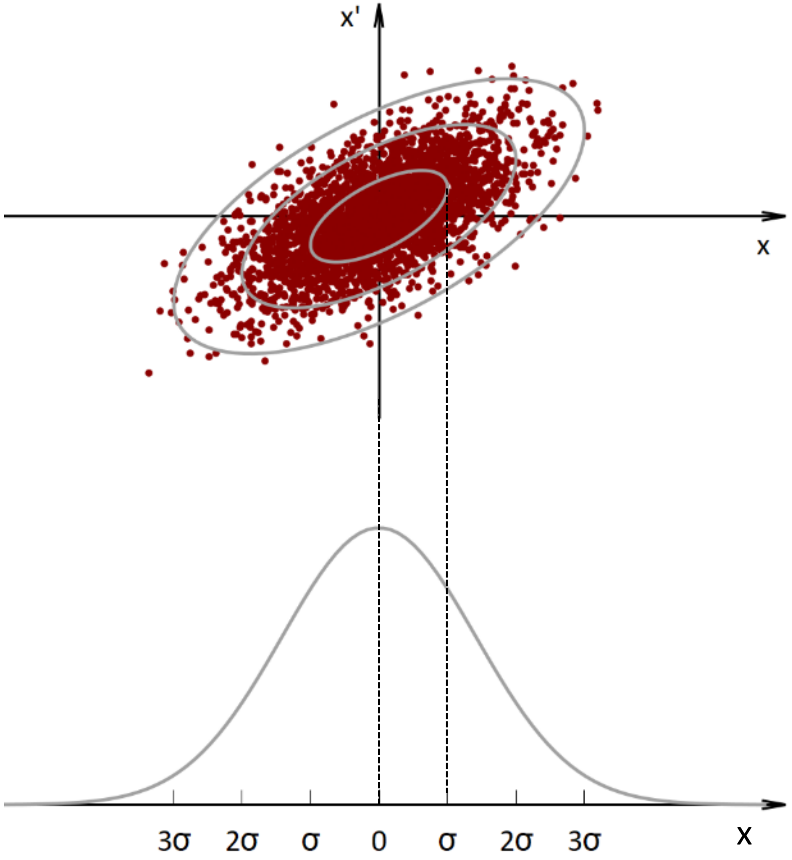
example value:
ε_n = 5 mm mrad

the geometric emittance at E_{min} is larger by a factor 20

the transverse beam size at E_{min} is larger by a factor $\sqrt{20} = 4.5$

→ magnet aperture and beam pipe must accommodate the beam at E_{min}

Recap: Beam Emittance



emittance characterizes the phase space volume of the beam, here of x,x'

beam emittance as statistical property:

$$\varepsilon_x = \sqrt{<x^2><x'^2> - <xx'>^2}$$

projected distribution:

$$\sigma_x = \sqrt{<x^2>}$$

$$\sigma_x = \sqrt{\varepsilon_x \beta_x}$$

Comparison to low energy cyclotron

the synchrotron is highly relativistic

the circulation time is practically constant

variations of f_{rev} as we considered for the cyclotron are not an issue

	cyclotron **E_k = 10MeV**	**synchrotron** **E_k = 25GeV**	**synchrotron** **E_k = 500GeV**
γ	**1.011**	**27.6**	**534**
β	**0.145**	**0.9993**	**0.999998**
ρ @ 2T	**0.229 m**		**835 m**

Step 3: Focusing using Quadrupoles

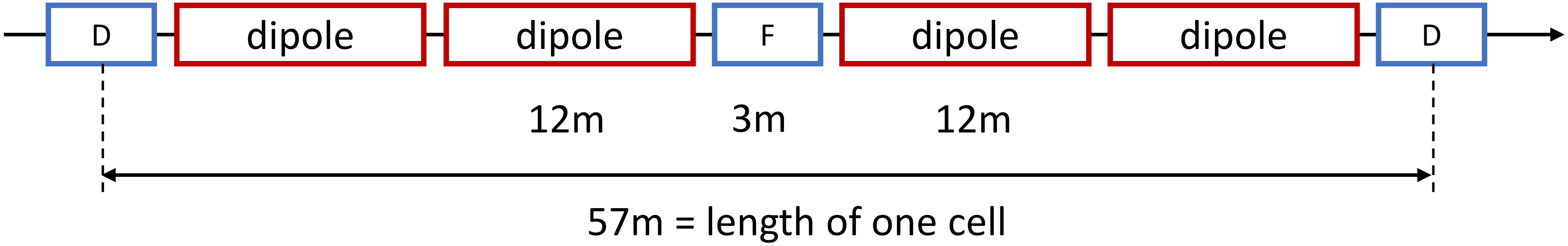
[SPS quadrupoles, A.Milanese, CERN]

FODO Cell Structure

Four dipoles per FODO cell: 440/4 = 110 cells

Besides bending also focusing is needed

→ part of the circumference cannot be covered with bending field, but is needed for gaps between magnets and for quadrupoles



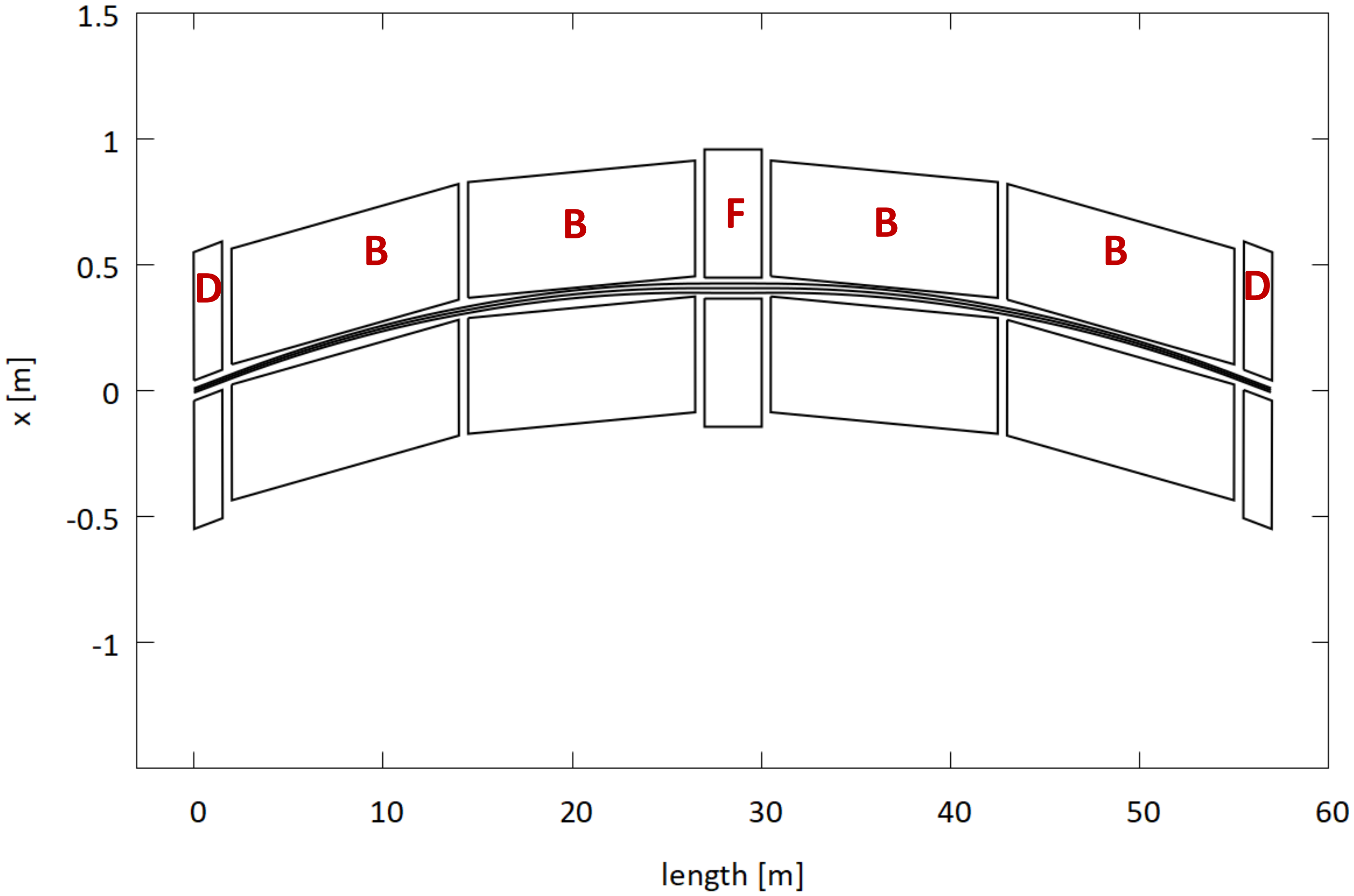
bending radius: $\rho = 840.4\text{m}$

average radius: $R = 57/48 \times \rho = 998\text{m}$

circumference: $C = 2\pi \times R = 6270\text{m}$

in practice more “insertions” needed: $C \approx 8000$ m

FODO Cell Geometry

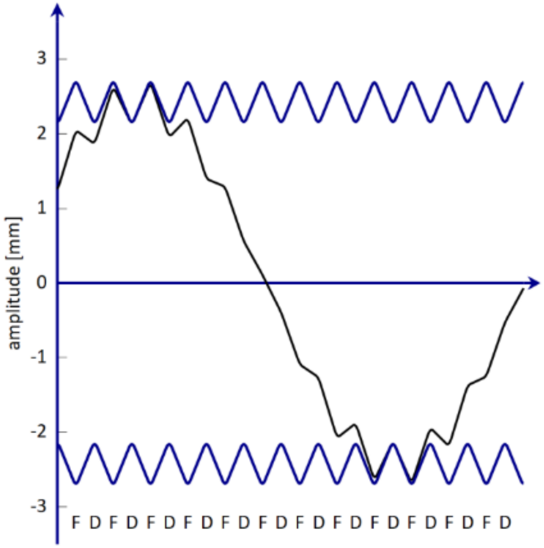


Note scale ratio s/x ≈ 20

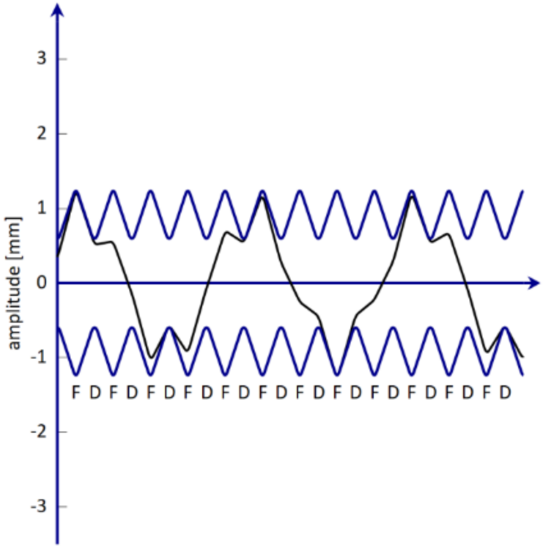
Recap: Beta Function

$$x(s) = A\sqrt{\beta(s)}\cos\varphi(s), \;\; \varphi(s) = \int_{t=s_0}^{s} \frac{dt}{\beta(t)}$$

→ the **beta function is a scaling factor** for the amplitude of orbit oscillations and their **local wavelength** at the same time



weak quads, β large



strong quads, β small

Rough estimates of key parameters

phase advance per cell:
we choose 90deg:

$$Q \approx N_{\text{cell}} \cdot \frac{90\text{deg}}{360\text{deg}} = 27.5$$

$$\boxed{Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)} \approx \frac{R}{\beta_{\text{avg}}}}$$

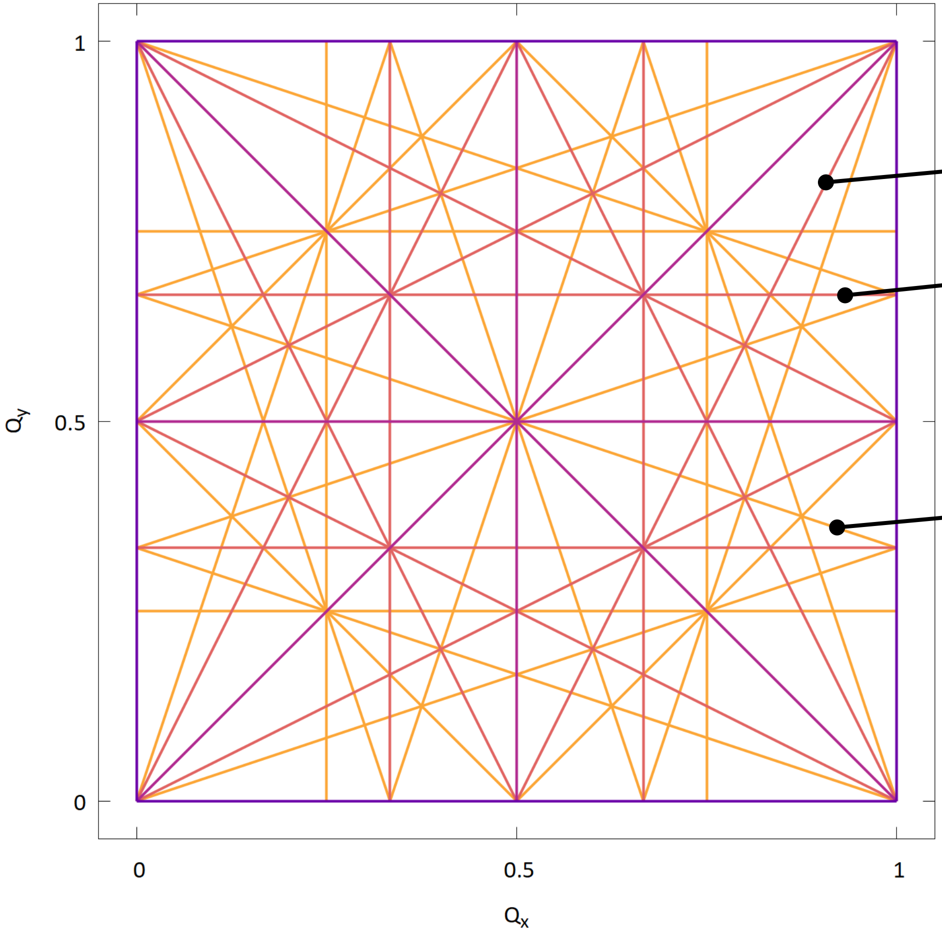
Beta-function is estimated from
smooth approximation (R=998m):

$$\beta_{\text{avg}} \approx R/Q = 36\,\text{m}$$

Resonance Plot

condition for resonance: $j \cdot Q_x + k \cdot Q_y = n$

order of resonance: $|j| + |k|$



$2Q_x - Q_y = 1$, order: 3

$3Q_y = 2$, order: 3

$Q_x + 3Q_y = 2$, order: 4

[up to 4-th order]

Estimate of dispersion, momentum compaction

$$D'' + K(s)D = \frac{1}{\rho}$$

simplifying assumptions:

$$D(s) = D_{\text{avg}} = \text{const}$$
$$K(s) = 1/\beta^2_{\text{avg}}$$
$$\beta_{\text{avg}} = R/Q$$

$$\boxed{D_{\text{avg}} \approx \frac{R}{Q^2}}$$

$$\boxed{\alpha_c \approx \frac{< D >}{R} = \frac{1}{Q^2}}$$

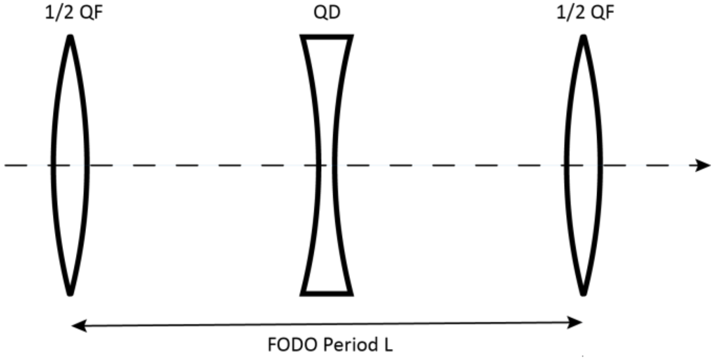
our 500GeV lattice, Q ≈ 27.5, R ≈ 998m:

$$D_{\text{avg}} = 1.32\,\text{m}$$
$$\alpha_c = 1.3 \times 10^{-3}$$

momentum compaction = path length vs momentum change

$$\frac{\Delta C}{C} = \alpha_c \frac{\Delta p}{p}$$

Recap: FODO Cell



thin lens approximation:

$$\boldsymbol{M}_{\mathrm{FODO}} = \begin{pmatrix} 1-\frac{L^2}{8f^2} & L\left(1+\frac{L}{4f}\right) \\ -\frac{1}{f^*} & 1-\frac{L^2}{8f^2} \end{pmatrix}, \quad \frac{1}{f^*} = \frac{L}{4f^2}\left(1-\frac{L}{4f}\right)$$

F: focusing quadrupole
D: defocusing quadrupole

f = focal length of quadrupole

Simplest case: $f = f_D = - f_F$, $l_1 = l_2$ (distance between quads)

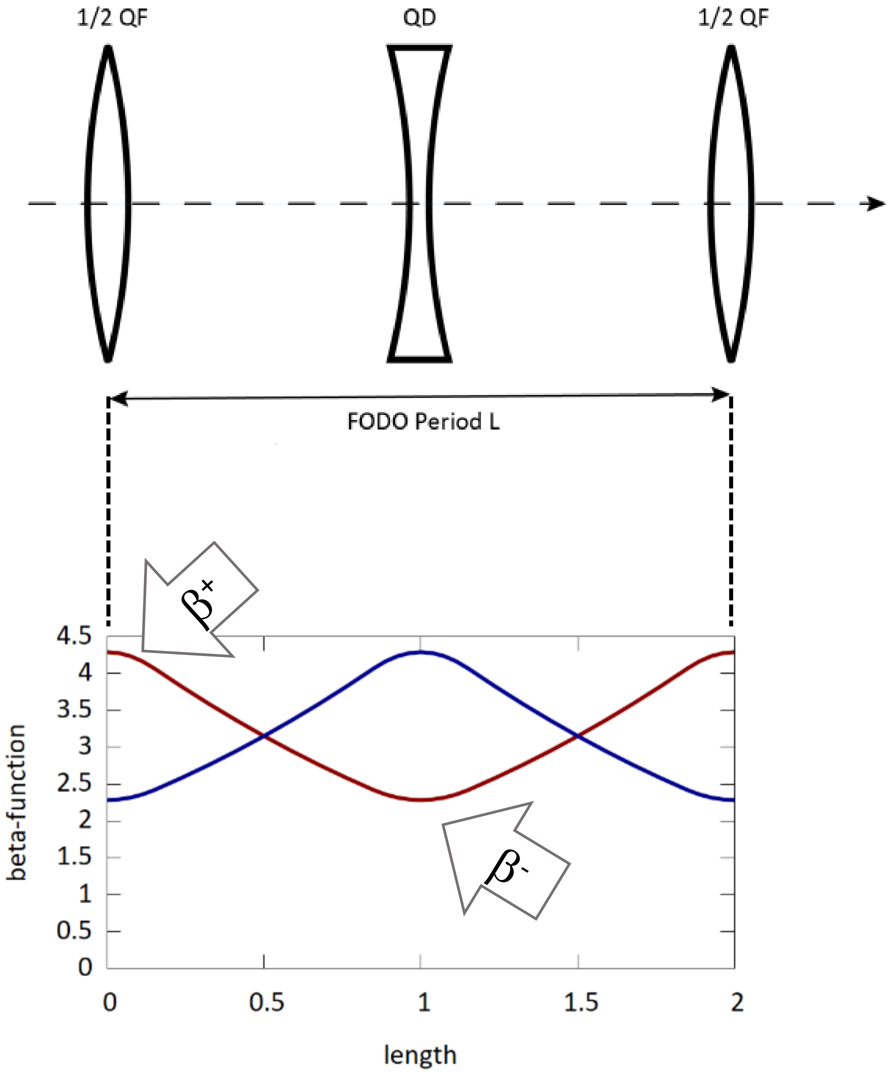
Recap: FODO Cell Parameters

we obtain for β^+ in the focusing quad and β^- in the defocusing:

$$\beta^{\pm} = L\,\frac{1 \pm \sin(\mu/2)}{\sin\mu}$$

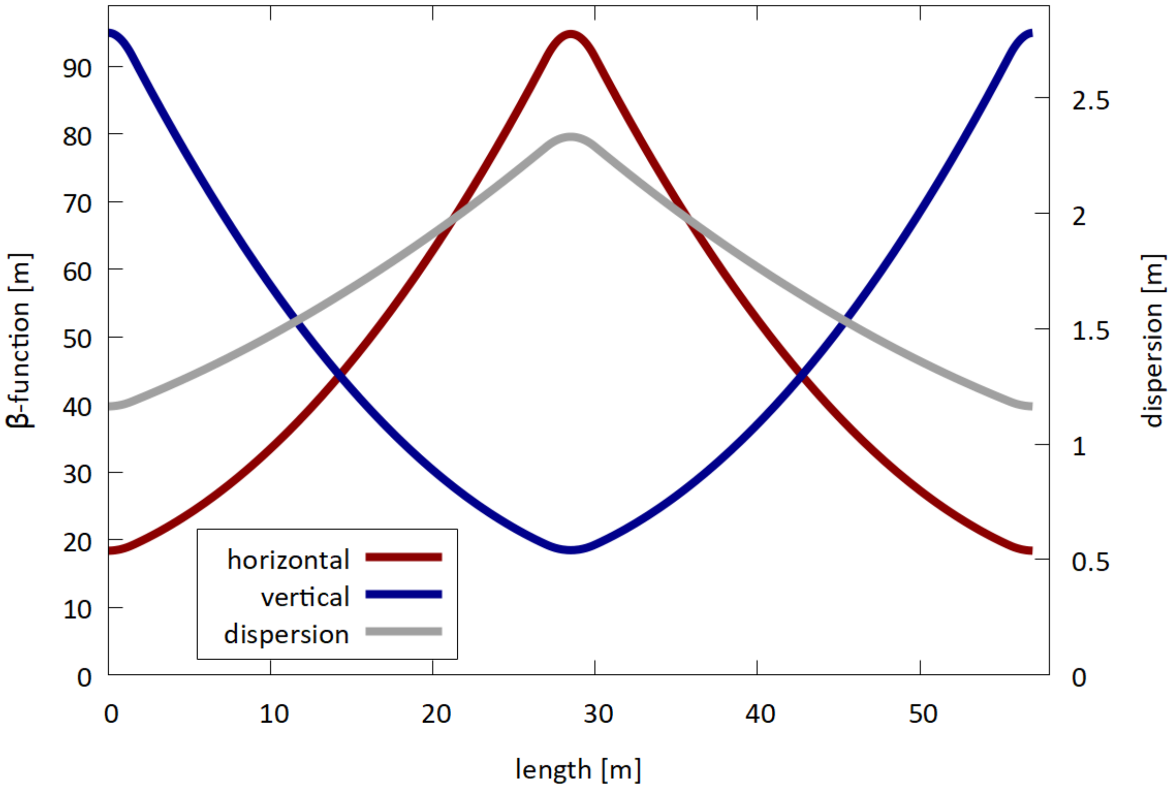
phase advance per cell (adjustable):

$$\sin(\mu/2) = \frac{L}{4f}$$



FODO Cell Beam Optics (90deg)

the three main optics functions in the FODO cell are β_x, β_y, D_x



$$\beta^{+} \approx 92\,\mathrm{m}$$

$$\beta^{-} \approx 18\,\mathrm{m}$$

$$\beta_{\mathrm{avg}} \approx \sqrt{\beta^{+}\,\beta^{-}} = 40.5\,\mathrm{m}$$

(estimate was 36.3m)

Emittance and Beam Size

normalized emittance ↔ geometric emittance

$$\varepsilon_n = 5\,\mathrm{mm\,mrad} \;\rightarrow\; \varepsilon_{\,500\,\mathrm{GeV}} = 10^{-8}\,\mathrm{mm\,mrad}$$

$$\varepsilon_{\,25\,\mathrm{GeV}} = 2 \cdot 10^{-7}\,\mathrm{mm\,mrad}$$

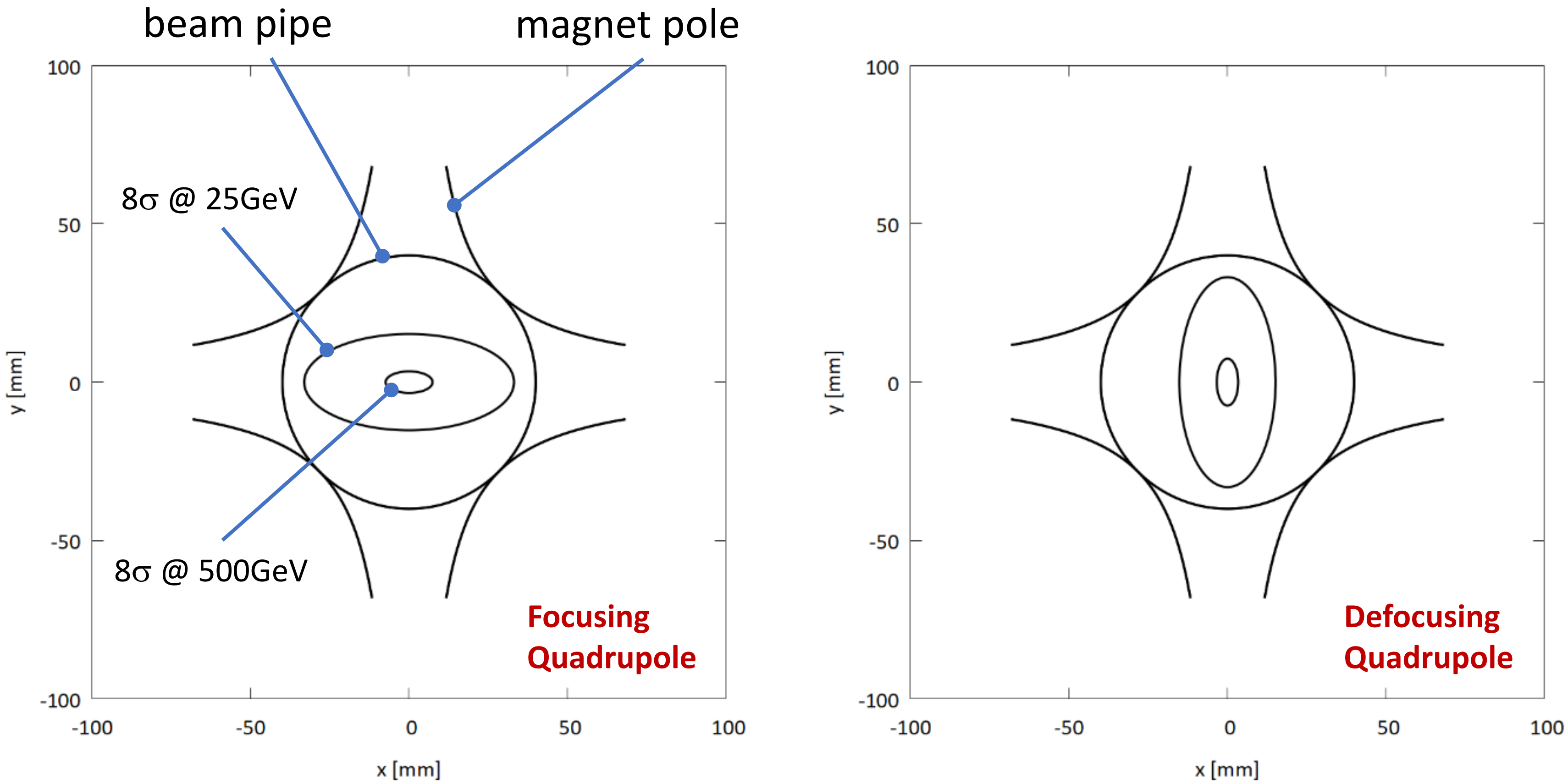
Beam size and angular spread:

$$\sigma_x = \sqrt{\varepsilon \beta_x},\;\; \sigma'_x = \sqrt{\varepsilon / \beta_x}$$

max/min values in FODO lattice:

	σ500	σ'500	σ25	σ'25
β^+ = 92m	1.0 mm	10 μrad	4.3 mm	47 μrad
β^- = 18m	0.4 mm	23 μrad	1.8 mm	102 μrad

Beam Envelopes, at 25GeV and at 500GeV

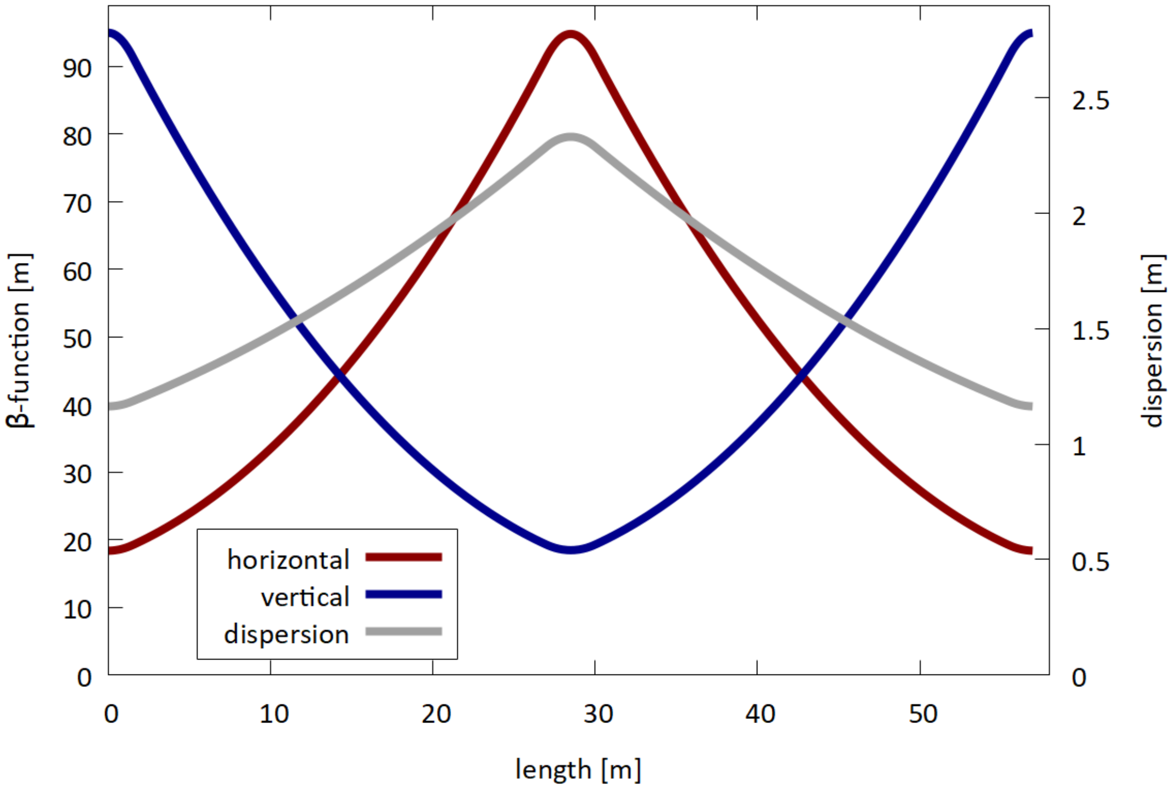


The **normalised emittance** is invariant during acceleration: $\varepsilon_n = \beta\gamma\varepsilon$

$$\sigma_x = \sqrt{\beta_x \varepsilon_x}$$
$$\sigma_y = \sqrt{\beta_y \varepsilon_y}$$

norm. emittance:
ε_n = 5 mm mrad

FODO Cell Dispersion (90deg)



$$D^{+} \approx 2.3\,\mathrm{m}$$

$$D^{-} \approx 1.2\,\mathrm{m}$$

$$D_{\mathrm{avg}} \approx \sqrt{D^{+}\,D^{-}}$$

$$= 1.6\,\mathrm{m}$$

(estimate 1.3m)

Recap: Chromaticity

Chromaticity ξ = change of tune per relative change of momentum:

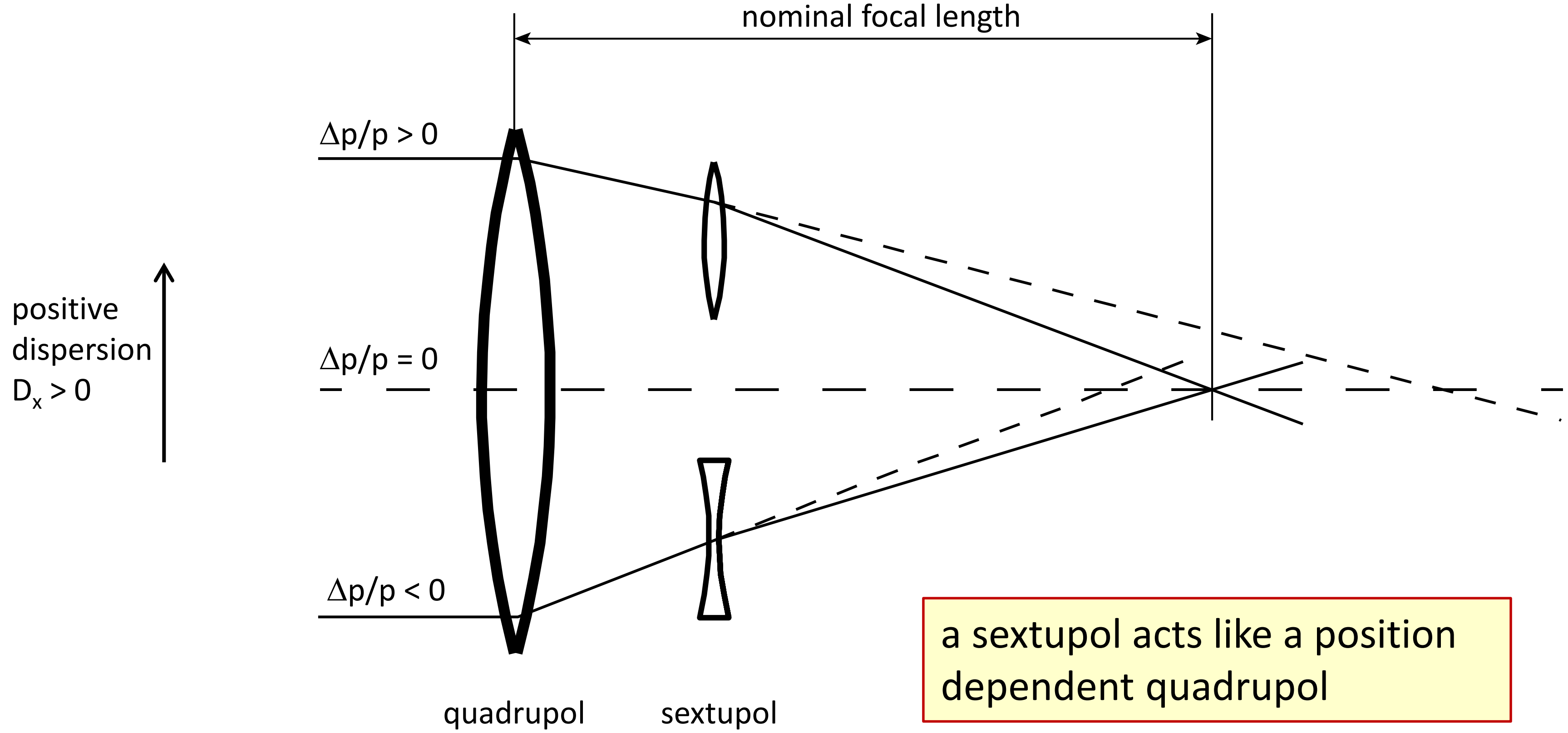
$$\Delta Q = \xi \, \frac{\Delta p}{p_0}$$

integration over gradients around ring, betafunction as “sensitivity factor”:

$$\boxed{\xi_x = -\frac{1}{4\pi} \oint K(s) \beta_x(s) ds}$$

→ chromaticity is overall negative in a FODO lattice and it must be corrected towards a small number using sextupoles besides each quadrupole

Recap Chromaticity: Correction using Sextupoles



a sextupol acts like a position dependent quadrupol

total chromaticity in a ring:

$$\xi_{\text{tot}} = \frac{1}{4\pi} \oint (m(s)D(s) - K(s))\beta_x(s)ds$$

Summary Case Study Proton Accelerators

- a 10 MeV classical cyclotron with a simple s.c. magnet as cost effective solution for a hospital
- a 500GeV large proton synchrotron with 7km circumference, FODO structure and electron beam option

keywords:

cyclotron, relativistic factors β & γ*, magnetic rigidity, cyclotron frequency,*

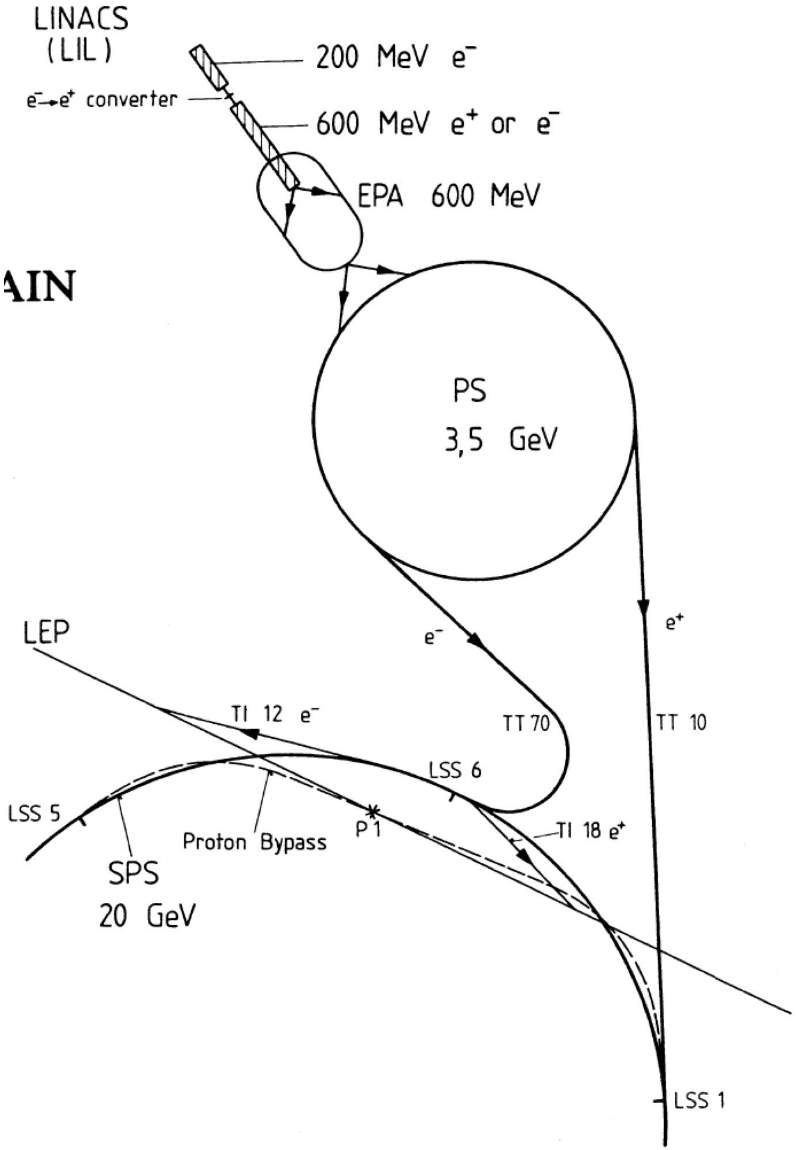
field index & tunes in cyclotron

synchrotron, transition energy, normalized & geometric emittance

beta function, FODO structure, phase advance per cell, betatron frequency/tune, chromaticity

How about electrons in same Synchrotron?

The CERN SPS was used to accelerate also Electrons and Positrons to ≈20GeV for LEP.



Our example synchrotron is good for 500GeV protons.
Why only 20GeV electrons?

Parameters for electron acceleration.

E_{min}	3.5 GeV
E_{max}	20 GeV
N_e (e^+ & e^-)	2×10^{11}
I	1.5 mA

note circulation time: 21μs

- for leptons the maximum energy results from SR losses and RF voltage/power
- the equilibrium emittance can be reduced with stronger focusing, SPS was operated with 135deg phase advance per FODO cell

20GeV → SR losses, RF voltage and power

energy loss / turn
per photon

$$\boxed{U_0[\mathrm{keV}] = 88.5 \times \frac{E^4[\mathrm{GeV}^4]}{\rho[\mathrm{m}]}}$$

= 17MeV / turn

our machine at 20GeV:

E_{max}	20 GeV
U_0	17 MeV
I	1.5 mA
P_{RF}	25 kW

hypothetical @ 500GeV:

E_{max}	500 GeV
U_0	6.6 TeV!
I	1.5 mA
P_{RF}	9.9 GW!

energy loss / turn →

beam current →

RF power →

Critical Energy of Dipole Radiation

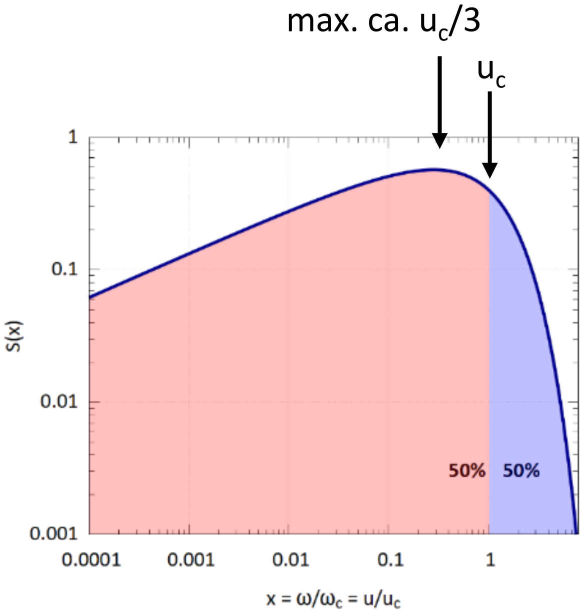
critical energy of radiation: $u_c\,[\mathrm{keV}] \approx 2.218\,\frac{E^3[\mathrm{GeV}^3]}{\rho[\mathrm{m}]}$

in our model accelerator (ρ=840m):

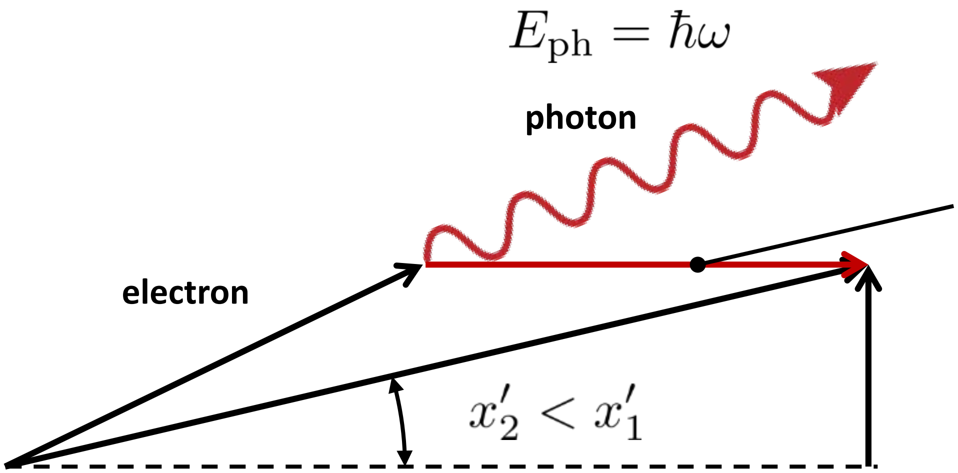
E_c(20 GeV) = 21 keV

E_c(3.5 GeV) = 0.11 keV

photon wavelength: $1\mathring{A} = 10^{-10}\,\mathrm{m} \sim 12.4\ \mathrm{keV}$



Radiation Damping



photon radiated in direction of motion

lost energy recovered by RF system

→ **angle reduced**
→ **emittance reduced**

damping times:
(most cases: D=0)

$$\boxed{\frac{1}{\tau_i} = \frac{U_0}{2\tau_0 E_s} \times J_i, \quad J_\epsilon = 2 + \mathcal{D}, \quad J_y = 1, \quad J_x = 1 - \mathcal{D}}$$

The balance between emission of photons with stochastic energy distribution and compensation of average energy loss leads to an equilibrium emittance.

Estimate of equilibrium electron beam emittance

$$\boxed{\varepsilon_x \approx \frac{C_q R}{J_x \rho_0} \frac{\gamma^2}{Q_x^3}}$$

energy

focusing

$$C_q = 3.84 \times 10^{-13}\,\mathrm{m}$$

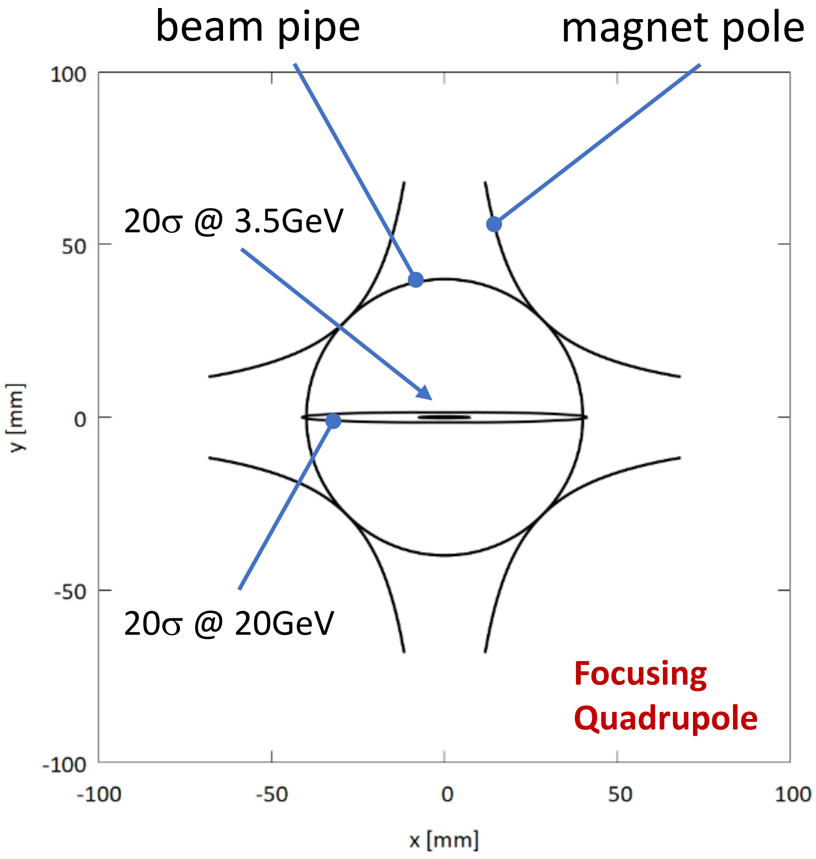
$$R/\rho_0 \approx 1.3$$

$$J_x \approx 1$$

Energy	Emittance
E = 20 GeV	$\varepsilon_x \approx$ 34 nm
E = 3.5 GeV	$\varepsilon_x \approx$ 1.1 nm

- for electrons the beam is larger at higher energies !
- vertical emittance: assume coupling factor 1/50 = 2% → beam is flat

Beam Envelopes for Electrons

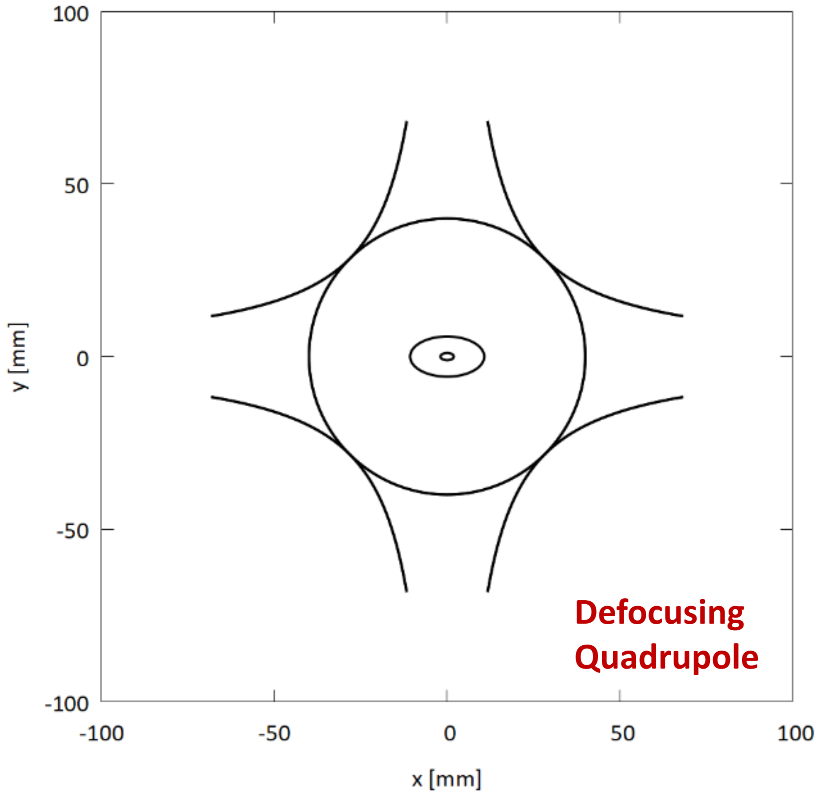

beam pipe

magnet pole

20σ @ 3.5GeV

20σ @ 20GeV

Focusing Quadrupole


Defocusing Quadrupole

Summary Electron Synchrotron Example

- although the magnets are good for 500GeV , electrons can reasonably be accelerated only to several 10GeV; limited by synchrotron radiation losses
- the electron beam assumes an equilibrium size which is larger at 20GeV than at 3.5GeV

key topics:

properties of radiation: synchrotron radiation loss per turn $\propto E^4$*, critical energy of radiated photons* $\propto E^3$*, photon energies in keV range, 12keV correspond to 1Å*

electron beam shrinks to equilibrium emittance = balance of radiation damping and quantum excitation, emittance scales as E^2 *and* Q_x^{-3}

Appendix: focusing in a classical cyclotron

centrifugal force mv²/r

Lorentz force qv×B

$$m\ddot{r} = mr\dot{\theta}^2 - qr\dot{\theta}B_z$$

focusing: consider small deviations *x* from beam orbit *R* (*r = R+x*):

$$\begin{aligned}
\ddot{x} + \frac{q}{m}vB_z(R+x) - \frac{v^2}{R+x} &= 0, \\
\ddot{x} + \frac{q}{m}v\left(B_z(R) + \frac{\mathrm{d}B_z}{\mathrm{d}R}x\right) - \frac{v^2}{R}\left(1 - \frac{x}{R}\right) &= 0, \\
\ddot{x} + \omega_{\mathrm{c}}^2(1+k)x &= 0.
\end{aligned}$$

$$\text{using: } \omega_{\mathrm{c}} = qB_z/m = v/R,\ r\dot{\theta} \approx v, k = \tfrac{R}{B}\tfrac{dB}{dR}$$

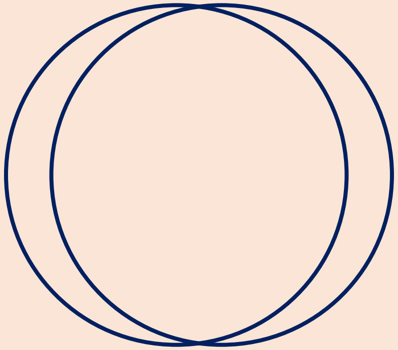
Appendix: betatron tunes in cyclotrons

thus in radial plane:

$$\begin{aligned}\omega_r &= \omega_c\sqrt{1+k} = \omega_c\nu_r \\ \nu_r &= \sqrt{1+k} \\ &\approx \gamma\end{aligned}$$

using isochronicity condition

note: simple case for *k = 0*: $\nu_r = 1$
(one circular orbit oscillates w.r.t the other)



using Maxwell to relate B_z and B_R:

$$\mathrm{rot}\,\vec{B} = \frac{dB_R}{dz} - \frac{dB_z}{dR} = 0$$

in vertical plane:

$$\nu_z = \sqrt{-k}$$

k<0 to obtain vertical focus.

thus: in classical cyclotron *k < 0* required for vert. focus;
however this violates isochronous condition *k =* γ^2*-1 > 0*