Transverse Dynamics ::
Lattice Imperfections and Hamilton Formalism

Laboratory for Particle Accelerator Physics, EPFL

EPFL

Transverse Dynamics - continued

practical questions to be answered:

- ✓ How to ensure bound motion of a particle beam?
- ✓ What are conditions for stability?
- ✓ Amplitude and frequency of particle oscillations?
- ✓ Statistical beam properties like beam width and angular spread?
- ✓ How to design magnet lattices (arrangements of dipoles and quads in a line)?
- What is the impact of **field errors in bending and focusing** magnets?
- What happens when the motion in **horizontal and vertical plane is coupled**?
- How can we **treat non-linear effects and coupling approximately** but using a systematic approach?

Recap: Hills Equation of Motion

$$
\begin{aligned}
x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' - ky &= 0
\end{aligned}
$$

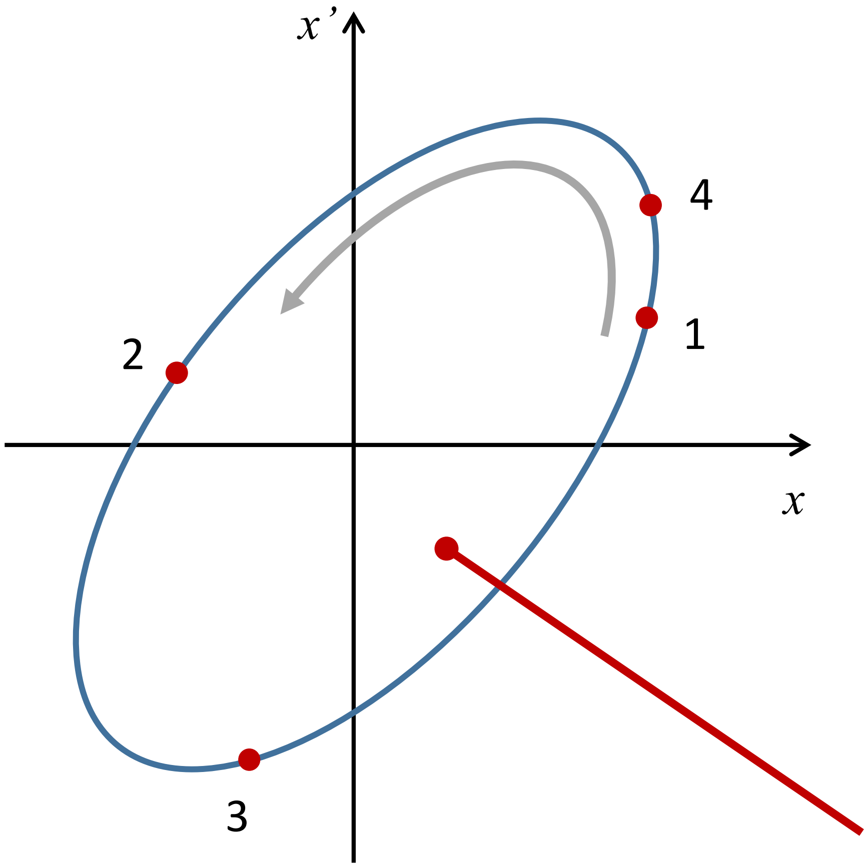
$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

DE is valid for

- drift spaces,
- quadrupoles (k≠0),
- combined function magnets (k≠0, 1/ρ≠0),
- off-momentum particles (Δp≠0, first order)

Phase Space Ellipse

[observing a particle at one location in a ring]



$$x(s) = \sqrt{2J\beta}\cos(\varphi)$$

$$x'(s) = -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right)$$

x, x' describe an ellipse in phase space when φ is varied

J = particle action (oscillation amplitude)

$$\text{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

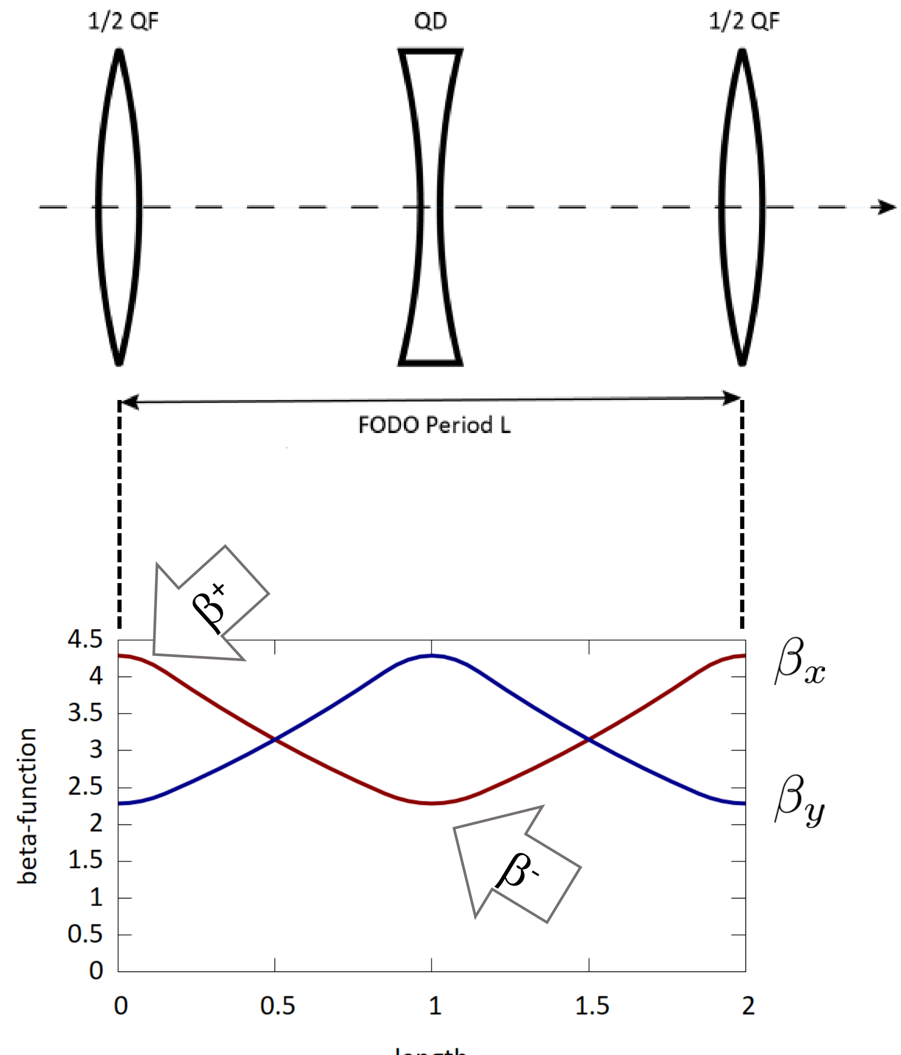
FODO Cell Parameters

we obtain for β^+ in the focusing quad and β^- in the defocusing:

$$\beta^{\pm} = L\,\frac{1 \pm \sin(\mu/2)}{\sin\mu}$$

phase advance per cell:

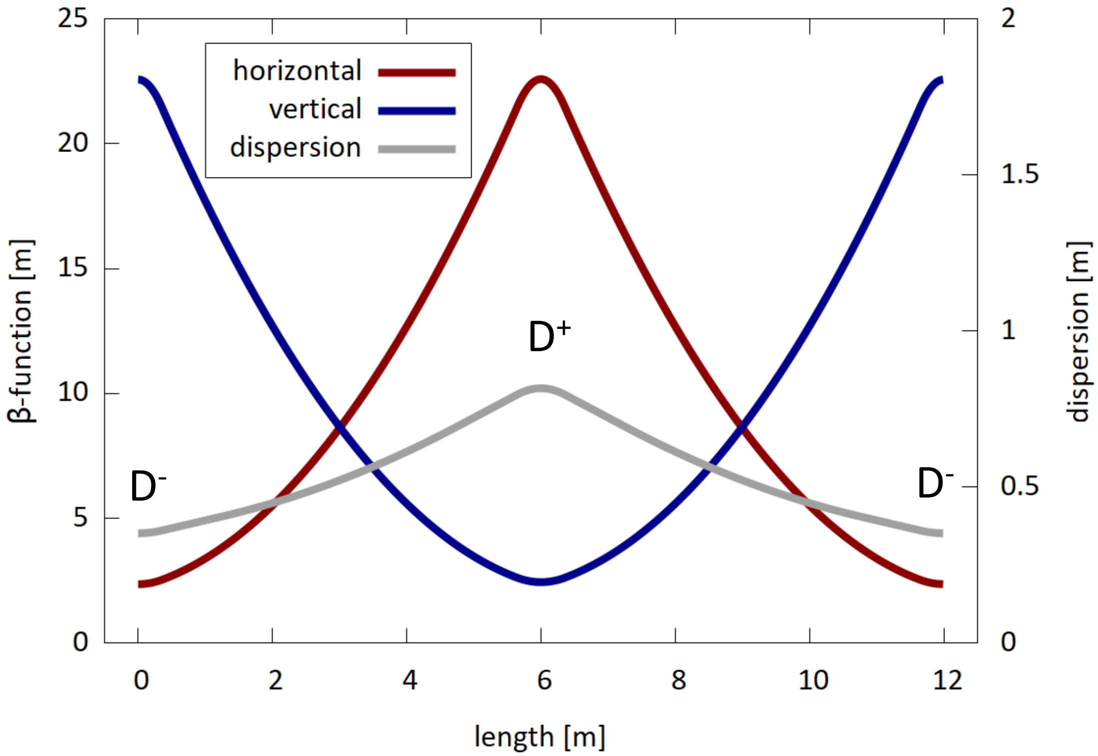
$$\sin(\mu/2) = \frac{L}{4f}$$



see Wiedemann sec. 10.1

FODO Cell with Dispersion

dispersion function D(s) is a periodic function in FODO cells with a maximum D^+ in a focusing quad and a minimum D^- in a defocusing quad



see Wiedemann sec. 10.2.3

Smooth Approximation – Dispersion, M.Compaction

$$D'' + K(s)D = \frac{1}{\rho}$$

simplifying assumptions:

$$D(s) = D_{\rm avg} = {\rm const}$$
$$K(s) = 1/\beta_{\rm avg}^2$$
$$\beta_{\rm avg} = R/Q$$
$$\rho = R$$

$$D_{\rm avg} \approx \frac{R}{Q^2}$$

$$\alpha_c \approx \frac{< D >}{R} = \frac{1}{Q^2}$$

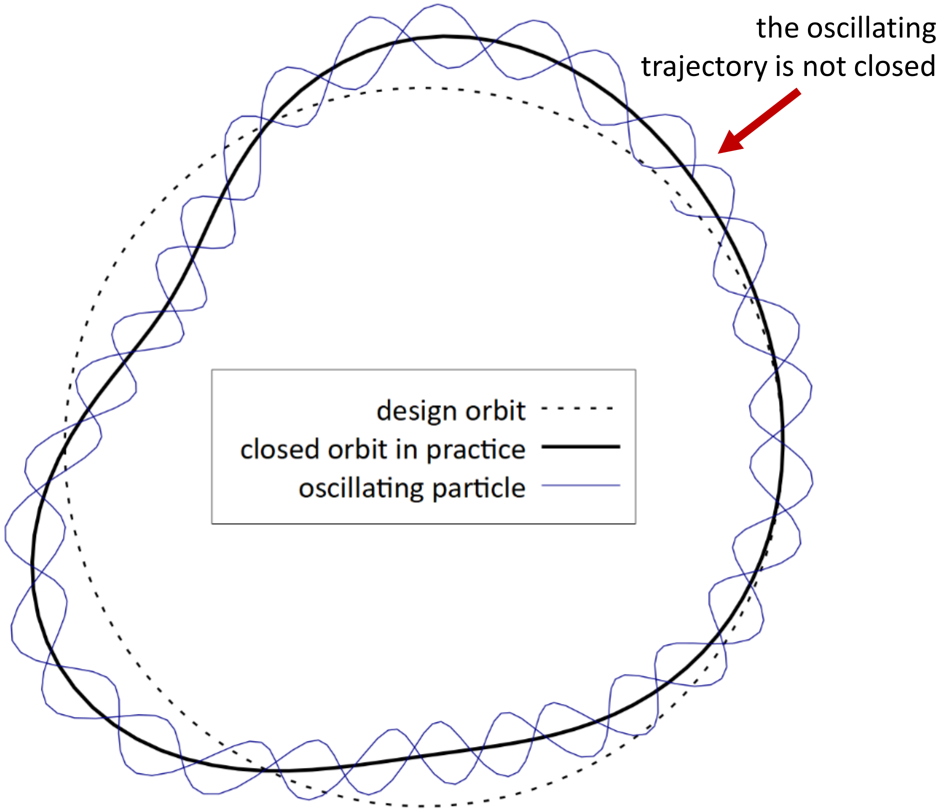
Next: Lattice Imperfections

- closed orbit distortion
- gradient errors

Closed Orbit

Closed Orbit =
Particle trajectory that closes on itself after one complete turn.

- in practice the **closed orbit** does not exactly follow the **design orbit**, but deviates due to small errors
- the closed orbit represents the beam center, **particles with nonzero actions oscillate around it**
- to assess practical implications and tolerances the effect of orbit distortions must be estimated



[closed orbit – conceptual sketch]

see Wiedemann sec. 15.2.3

Closed Orbit Distortion

- the desired bending of the beam by $1/\rho(s)$ is included in the calculation of the design orbit, particle on design orbit: $(x,x') = (0,0)$
- **here we consider an additional (erroneous) kick angle θ**
- this kick θ leads to an oscillation of the closed orbit around the design orbit

calculation by requesting a closed orbit:

$$\boldsymbol{M}\begin{pmatrix} x_0 \\ x_0' + \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$\rightarrow$ solve equation for x_0, x_0'

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \left(\boldsymbol{M}^{-1} - \boldsymbol{I}\right)^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix} \qquad \boldsymbol{M} = \begin{pmatrix} \cos 2\pi Q & \beta_0 \sin 2\pi Q \\ -\frac{1}{\beta_0}\sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$

Closed Orbit Distortion – Resulting Orbit

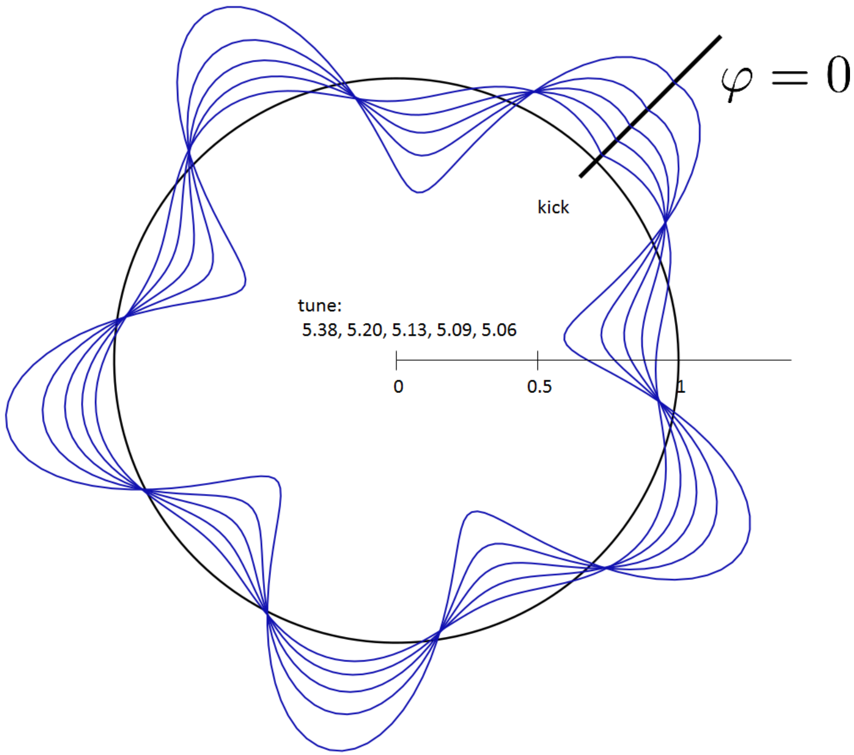
$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \left(\boldsymbol{M}_{\text{rev}}^{-1} - \boldsymbol{I}\right)^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix} \qquad \boldsymbol{M}_{\text{rev}} = \begin{pmatrix} \cos 2\pi Q & \beta_0 \sin 2\pi Q \\ -\frac{1}{\beta_0} \sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$

(here: α_0=0)

$$x_0 = \theta\beta_0 \, \frac{\cos(\pi Q)}{2\sin(\pi Q)}$$

$$x(s) = \theta \cdot \frac{\sqrt{\beta(s)\beta_0}}{2\sin(\pi Q)} \, \cos(\varphi(s) - \pi Q)$$

distorted orbits for varying Q,
solution explodes for Q → Integer



Orbit correction by applying additional kicks

single kick:

$$x(s) = \frac{\theta\sqrt{\beta(s)\beta_0}}{2\sin(\pi Q)}\ \cos(\varphi(s) - \pi Q)$$

kick θ is caused by an unwanted magnetic field, or an off-set quadrupole (errors)
however, kick(s) can be applied also on purpose to correct the orbit

with several kicks θ_j the contributions are added:

$$x_k = \sum_j \frac{\sqrt{\beta_k\beta_j}\ \cos(|\varphi_k - \varphi_j| - \pi Q)}{2\sin(\pi Q)}\ \theta_j$$

$$= \sum_j \boldsymbol{R_{k,j}}\theta_j \qquad \text{(= matrix multiplication)}$$

Orbit Correction

given is a set of beam positions representing an orbit x_k

calculate a set of (wanted) corrector strengths θ_j to minimize the orbit amplitude

this can be formulated as a problem of linear algebra (**R_{kj}** coefficients last slide):

$$\vec{x}_{\text{pos}} + \boldsymbol{R}\vec{\theta}_{\text{cor}} = 0$$

this is solved exactly for $N_{\text{pos}} = N_{\text{cor}}$, however in practice we need flexible solutions

Singular Value Decomposition (SVD) is one of many approaches:

$$\boldsymbol{R} = \boldsymbol{U} \cdot \boldsymbol{W} \cdot \boldsymbol{V}^{\text{T}}$$

$$\boldsymbol{R}_{\text{inv}} = \boldsymbol{V} \cdot \boldsymbol{W}^{-1} \cdot \boldsymbol{U}^{\text{T}}$$

W = diagonal matrix with singular values, inversion simple

solution:

$$\vec{\theta}_{\text{cor}} = -\boldsymbol{V}\boldsymbol{W}^{-1}\boldsymbol{U}^{\text{T}} \cdot \vec{x}_{\text{pos}}$$

- $N_{\text{pos}} = N_{\text{cor}}$: exact solution
- $N_{\text{pos}} < N_{\text{cor}}$: minimizes $|\vec{\theta}|$ (magnet currents)
- $N_{\text{pos}} > N_{\text{cor}}$: minimizes $|\vec{x}|$ (rms orbit deviation)

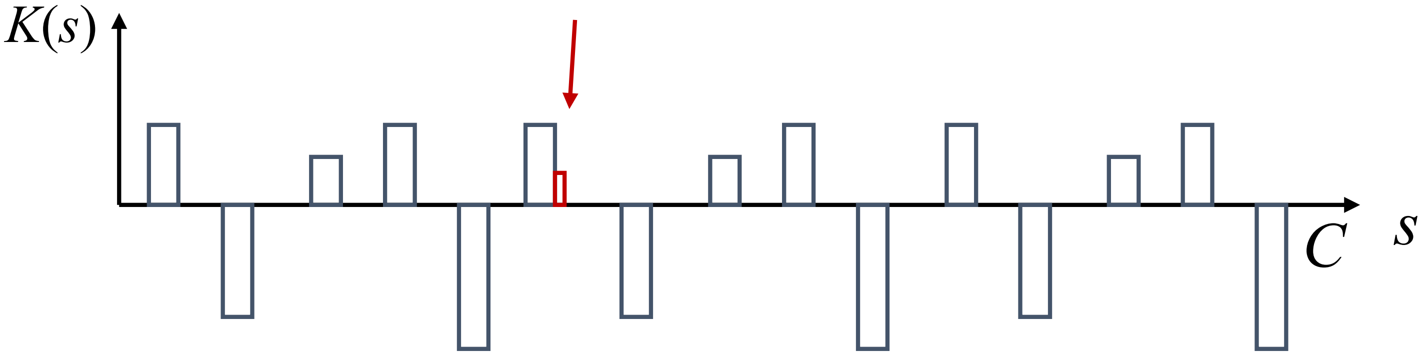
→ **in practice this is done using computer codes with many variants of algorithms**

see Wiedemann sec. 15.4.2

Gradient Error

$$x'' + (K_0(s) + \Delta K(s))\, x \quad = \quad 0$$

most simple case: the distortion of the gradient is short and can be treated as a thin lens



we want to know:

1. the tune shift caused by the error
2. the modification of the beam width (via computing $\Delta\beta(s)$)

Gradient Error – Tune Shift

method: modify 1-turn transport matrix by multiplying thin lens error matrix

$$\boldsymbol{M}_{\text{err}} = \boldsymbol{m}_{\text{err}}\ \boldsymbol{M}_0,\ \ \boldsymbol{M}_0 = \boldsymbol{I}\cos(\mu_0) + \boldsymbol{J}\sin(\mu_0)$$

add erroneous slice

$$\boldsymbol{m}_{\text{err}} = \begin{pmatrix} 1 & 0 \\ \Delta K l & 1 \end{pmatrix}$$

$$\boldsymbol{M}_{\text{err}} = \begin{pmatrix} 1 & 0 \\ \Delta K l & 1 \end{pmatrix} \cos(\mu_0) + \begin{pmatrix} \alpha & \beta \\ -\alpha \Delta K l - \gamma & -\beta \Delta K l - \alpha \end{pmatrix} \sin(\mu_0)$$

$$\cos(\mu) = \frac{1}{2}\text{Tr}\boldsymbol{M} = \cos(\mu_0) - \frac{1}{2}\beta \Delta K l \sin(\mu_0) \qquad \cos\mu \approx \cos\mu_0 - \Delta\mu \sin(\mu_0)$$

resulting tune shift for distributed gradient errors:

$$\Delta Q = \frac{1}{4\pi} \oint \beta(s) \Delta K(s) ds$$

see Wiedemann sec. 15.3.1

Gradient Error - Betafunction

similar derivation without proof:

$$\Delta\beta(s) = \frac{\beta(s)}{2\sin(2\pi Q)} \oint dt\ \beta(t)\Delta K(t) \cos\left(2(\varphi(s) - \varphi(t) - \pi Q)\right)$$

solution explodes for Q $\rightarrow$ Integer × 0.5

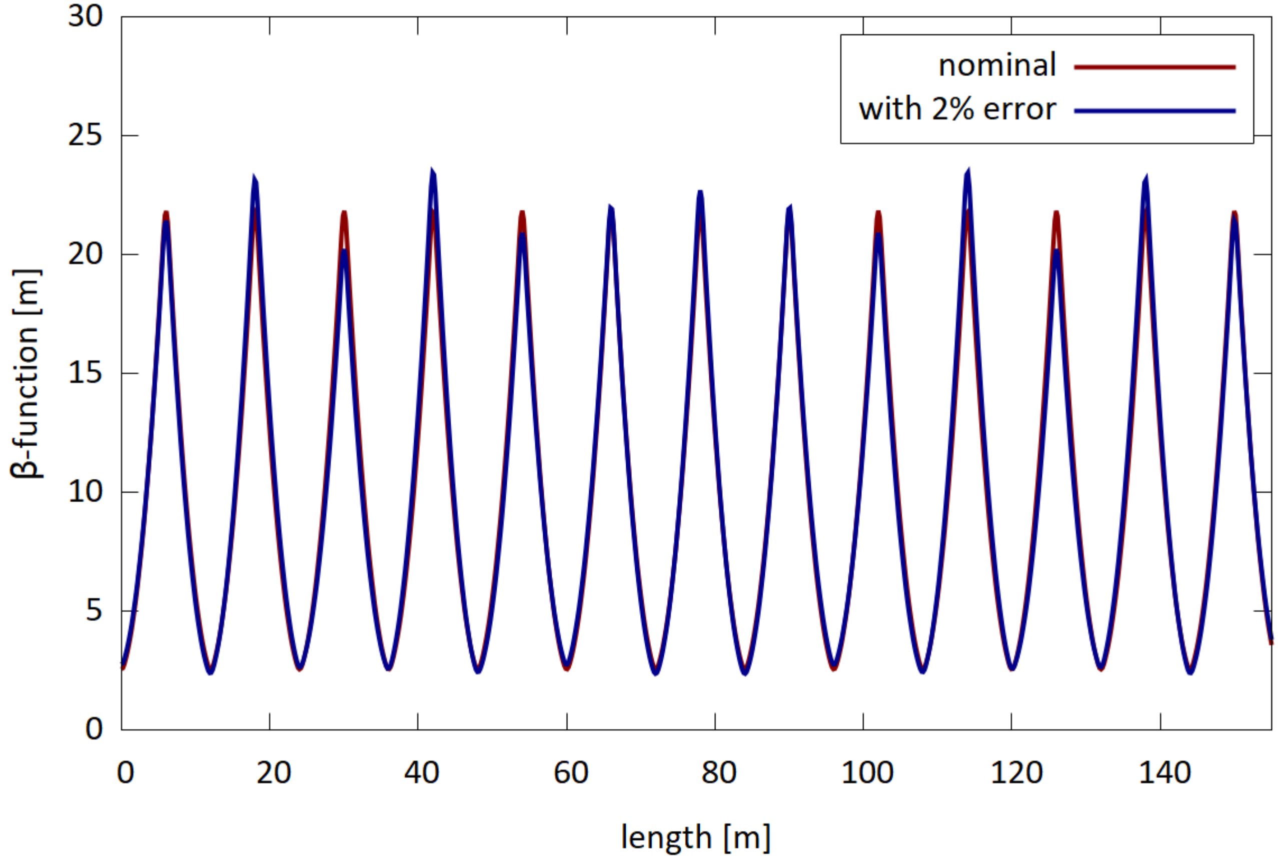
note: double frequency

- this error modulates the beam width around the ring
- the effect is called **„Beta-Beat“**
- the Beta-Beat propagates at the double frequency of an orbit distortion

see Wiedemann sec. 15.3.4

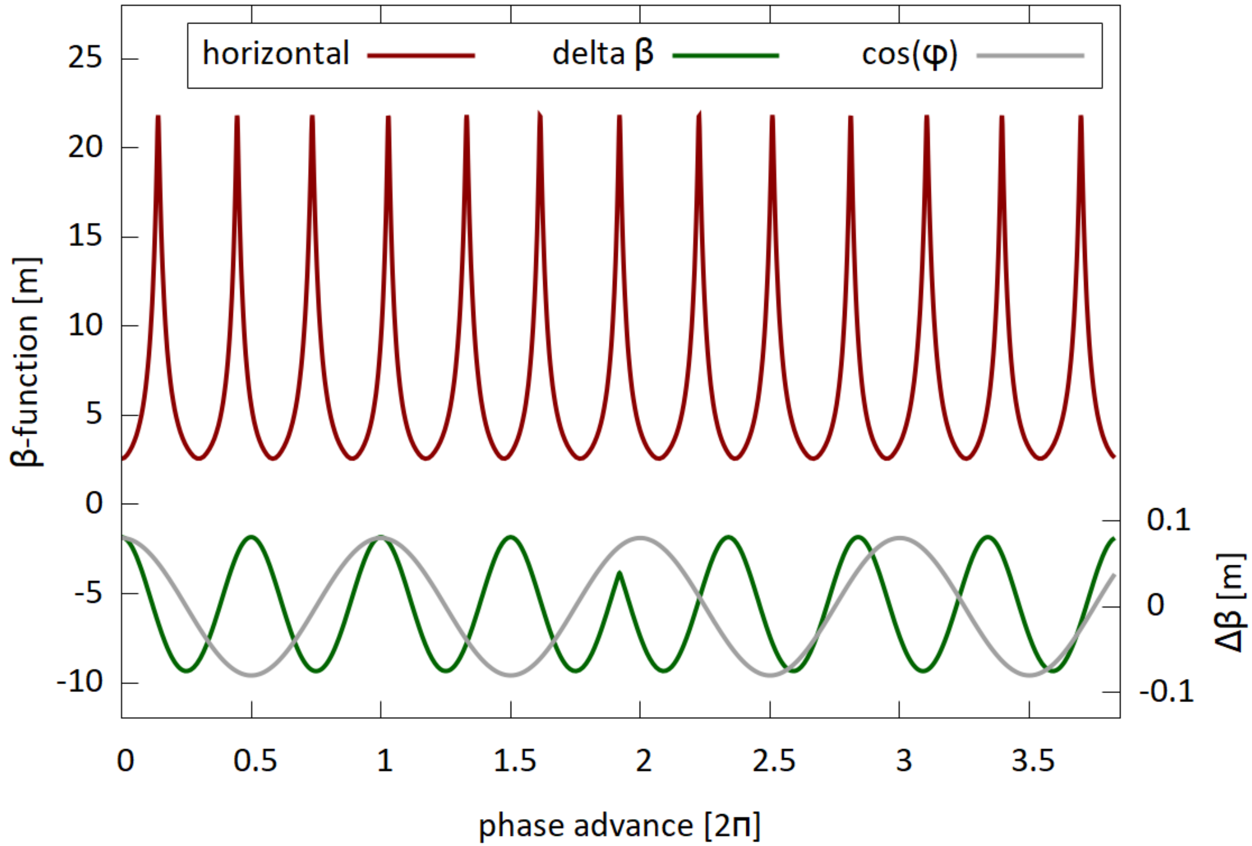
Gradient Error Example

accelerator lattice with 13 regular FODO cells, one quad in center has an error of +2%
the β function is modulated by ≈10% (beat frequency of 2x is not recognizable in graph)



Gradient Error Example continued

accelerator lattice with 13 regular FODO cells, one quad in center has an error of +2%
when $\Delta\beta/\beta$ is plotted against phase advance we see the “error kick” and the double beat frequency



Next: Coupling of Transverse Planes

- tilted quadrupole, origin of coupling
- stability criterion in presence of coupling
- solution of coupled equations
- Literature: Wiedemann Chap 20, Conte/McKay Chap 10.2

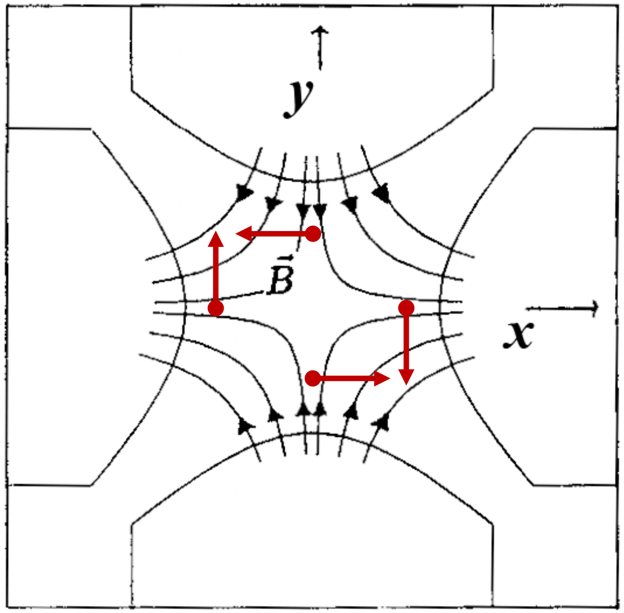
The tilted quadrupole couples x and y

a pure tilted quadrupole results from a normal quadrupole that is tilted by 45 degrees

$$F_x \propto -y$$

$$F_y \propto -x$$

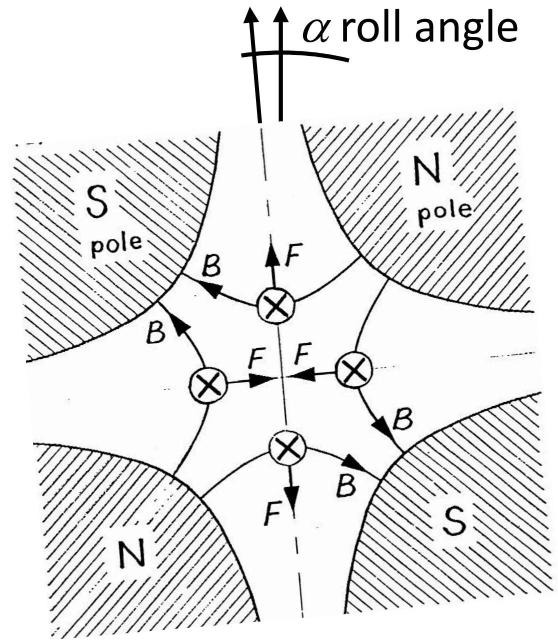
the force depends on the coordinate in the other plane



Coupling Errors in a Real Accelerator

a small **roll angle α of a normal quadrupole** produces a tilted quadrupole component

such errors are unavoidable, and their consequences must be assessed



$$\boldsymbol{M}_{\text{rolled}} = \boldsymbol{R}(-\alpha) \cdot \boldsymbol{M}_q \boldsymbol{R}(\alpha)$$

$$\boldsymbol{M}_{\text{rolled}} \approx \boldsymbol{M}_q + \alpha \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2/f & 0 \\ 0 & 0 & 0 & 0 \\ -2/f & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{R} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

thin lens approximation and small roll angle α

Stability Conditions for a Coupled Ring

„normal“ horizontal block 2x2

block w. coupling 2x2, often sparse

4x4 coupled transport matrix:

$$\boldsymbol{T} = \left(\begin{array}{c|c} \boldsymbol{M} & \boldsymbol{n} \\ \hline \boldsymbol{m} & \boldsymbol{N} \end{array}\right) \quad \text{again :} \quad \det \boldsymbol{T} = 1$$

analysing eigenvectors and –values:

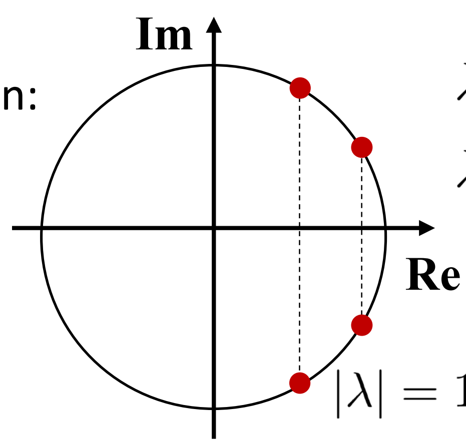
$$\boldsymbol{T}\vec{v_j} = \lambda_j \vec{v_j}, \;\; j = 1 \ldots 4 \quad \text{and :} \quad \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 1$$

as it turnes out, the EV‘s come in reciprocal pairs:

$$\lambda_{\mathrm{I}} = 1/\lambda_{-\mathrm{I}}, \;\; \lambda_{\mathrm{II}} = 1/\lambda_{-\mathrm{II}}$$

uncoupled: I $\equiv x$, II $\equiv y$, is a special case

EV`s for stable motion:



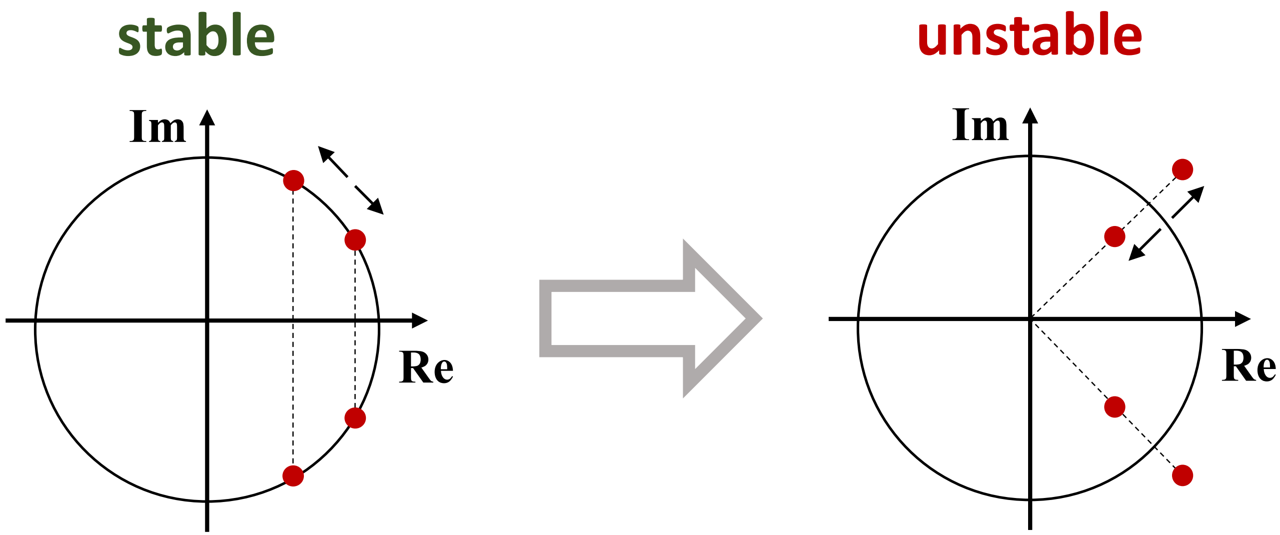
$$\lambda_{\mathrm{I},-\mathrm{I}} = e^{\pm i\mu_I}$$

$$\lambda_{\mathrm{II},-\mathrm{II}} = e^{\pm i\mu_{II}}$$

$$|\lambda| = 1, \;\; \text{unit circle}$$

see also „symplectic condition“ in Appendix

transition stable to unstable is possible for $Q_x \approx -Q_y$ (= Sum Resonance)



in presence of coupling sources:
EVs cannot be moved together, instead they move away from unit circle

EVs for unstable motion:

$$\lambda_{\mathrm{I},-\mathrm{I}} = r^{\pm 1} e^{\pm i \mu_I}$$

$$\lambda_{\mathrm{II},-\mathrm{II}} = r^{\pm 1} e^{\pm i \mu_{II}}$$

$$\rightarrow \lambda^n \text{ diverges}$$

sketch of EV calculation in coupled condition

resonance condition: $Q_1 \pm Q_2 \approx n;\ \mu = 2\pi Q$ ± for Sum/Diff resonance

→ use 4x4 one-turn transfer matrix with one quad rolled by small angle α, focal length f

from matrix calculation: $\kappa_{I,II} = \lambda_{I,II} + 1/\lambda_{I,II} \approx 2\cos\mu \pm 2\delta_{S,D}\sin\mu$

$$\delta_{S,D} = \begin{cases} i\sqrt{\beta_x\beta_y}\alpha/f & \text{for Sum resonance} \\ \sqrt{\beta_x\beta_y}\alpha/f & \text{for Difference resonance} \end{cases}$$

β_1,β_2: optics functions at quad, f: focal length of rolled quad, α: small roll angle

→ this is a recipe to calculate all four Eigenvalues in presence of coupling

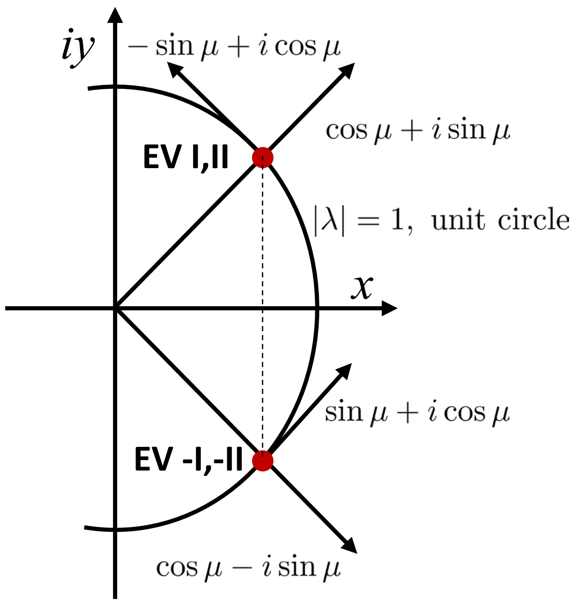
(without coupling these would degenerate into just two EVs):

$$\lambda_{I,-I,II,-II} = e^{\pm i\mu} \pm^{(I:II)} \delta_{S,D}\,(\sin\mu \mp i\cos\mu)$$

note that $\delta_{S,D}$ can be imaginary or real depending on Sum/Diff resonance

coupled EVs continued …

the EVs move apart from the degenerate I,II values ($Q_x = \pm Q_y$), depending on the nature of the resonance



$$\lambda_I = e^{i\mu} + \delta_{S,D}\,(\sin\mu - i\cos\mu)$$

$$\lambda_{II} = e^{i\mu} - \delta_{S,D}\,(\sin\mu - i\cos\mu)$$

$$\lambda_{-I} = e^{-i\mu} + \delta_{S,D}\,(\sin\mu + i\cos\mu)$$

$$\lambda_{-II} = e^{-i\mu} - \delta_{S,D}\,(\sin\mu + i\cos\mu)$$

$$\delta_{S,D} = \begin{cases} i\sqrt{\beta_1\beta_2}\,\alpha/f & \text{for Sum resonance} \\ \sqrt{\beta_1\beta_2}\,\alpha/f & \text{for Diff. resonance} \end{cases}$$

for Sum resonances the EVs move apart radially $\rightarrow$ **unstable**
for Difference resonances the EVs separate, but stay on unit circle $\rightarrow$ **stable**

discussion: sum and difference resonance

so far discussed: treatment in matrix formalism and via Eigenvalues, also possible and more general is treatment through **perturbation theory in Hamilton formalism**

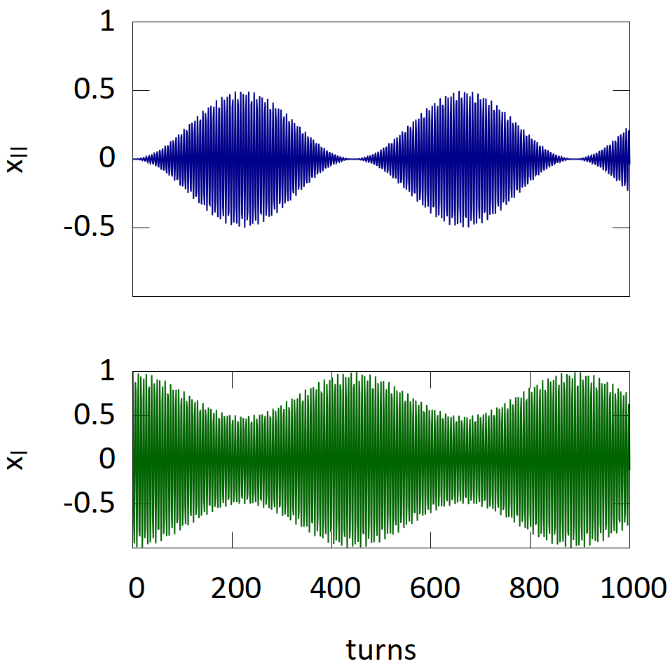
conserved for **sum resonance: J_I-J_{II} = const.**, each action can grow indefinitely: unstable

conserved for **diff. resonance: J_I+J_{II} = const.** : stable motion but exchange of action

<u>difference resonance</u>

particle oscillation started with $J_I \neq 0, J_{II} = 0$

action is exchanged as known for coupled pendulums



$$J_1(\theta) = \frac{A}{4\Omega^2}\left(\Delta^2 + \kappa^2 \cos^2(\Omega\theta)\right)$$

$$J_2(\theta) = \frac{A}{4\Omega^2}\left(\kappa^2 \sin^2(\Omega\theta)\right)$$

$$J_1(0) \neq 0;\ \ J_2(0) = 0$$

$$\Delta = Q_1 - Q_2 - n$$

$$\Omega = \frac{1}{2}\sqrt{\kappa^2 + \Delta^2}$$

$$J_1 + J_2 = A = \mathrm{const}$$

discussion: difference resonance

for the difference resonance the motion is stable, but tunes cannot be moved together

two oscillation modes I,II are observed instead of the uncoupled x,y modes

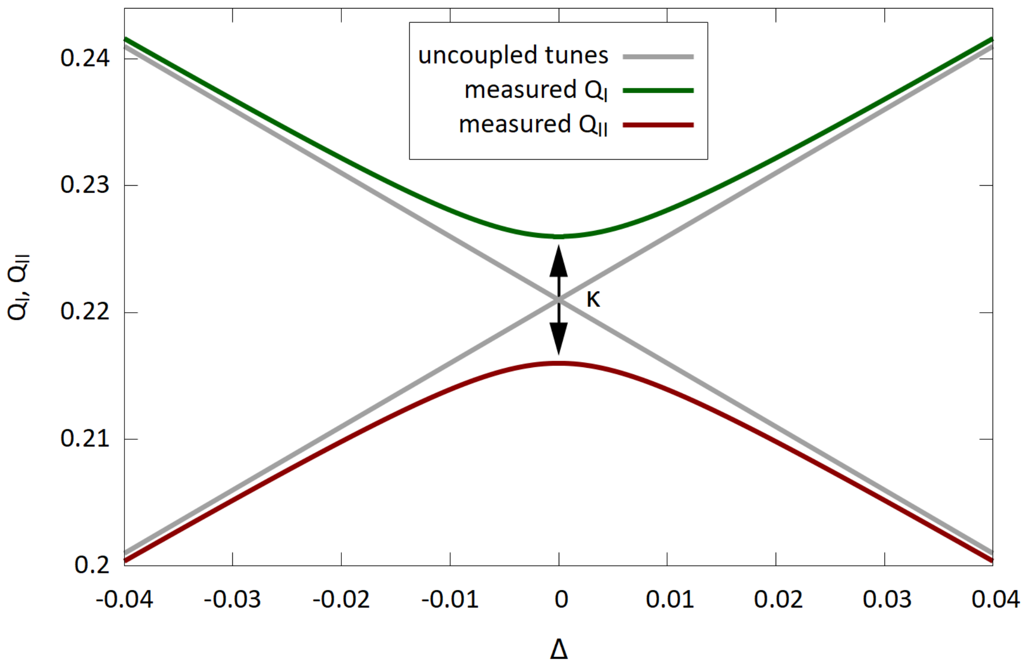
κ is a measure of the strength of coupling, i.e. roll angle of quads and other sources

scan of one planes tune

observed are two modes whose frequencies stay separated

$$Q_{I,II} = \frac{1}{2}(Q_x + Q_y) \pm \frac{1}{2}\sqrt{\Delta^2 + \kappa^2}$$

$$\Delta = Q_x - Q_y$$



Next: Hamiltonian Formalism and Perturbation Theory

- Hamiltonian Formalism: What is it and why do we use it?
- Introduction by Pendulum example
- Hamiltonian for Accelerators in (x,x') and (J,ψ)
- Perturbation treatment for Quad Error, Coupling, Sextupole

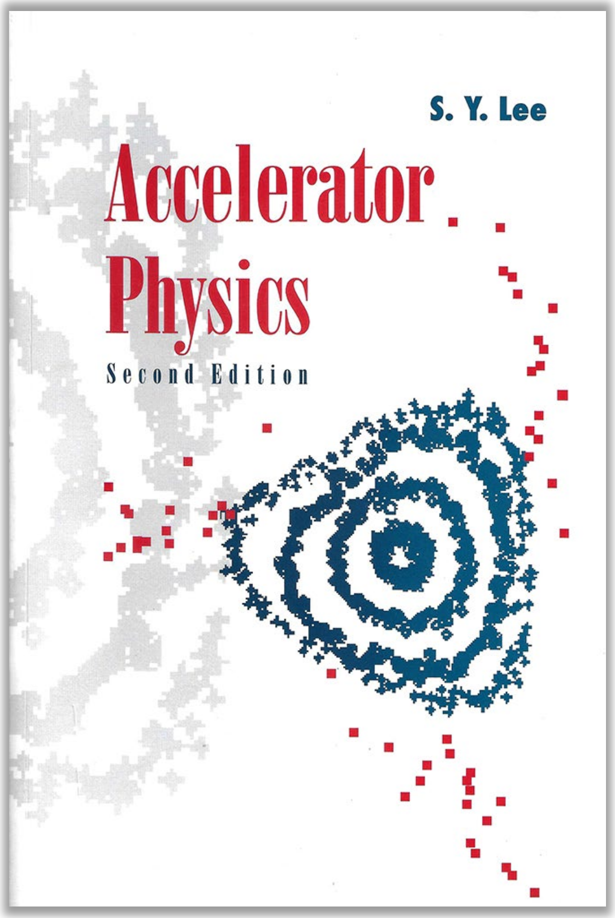
EPFL

Introductory Remarks

Nonlinear resonances is one of the top subjects in accelerator physics. It concerns the long term stability of particles in a storage ring.

some aspects for ring design:

- resonant tune values
- nonlinear driving terms in general
- resonance overlap and chaotic motion
- sextupoles for chromatic correction
- octupoles for detuning of betatron oscillations



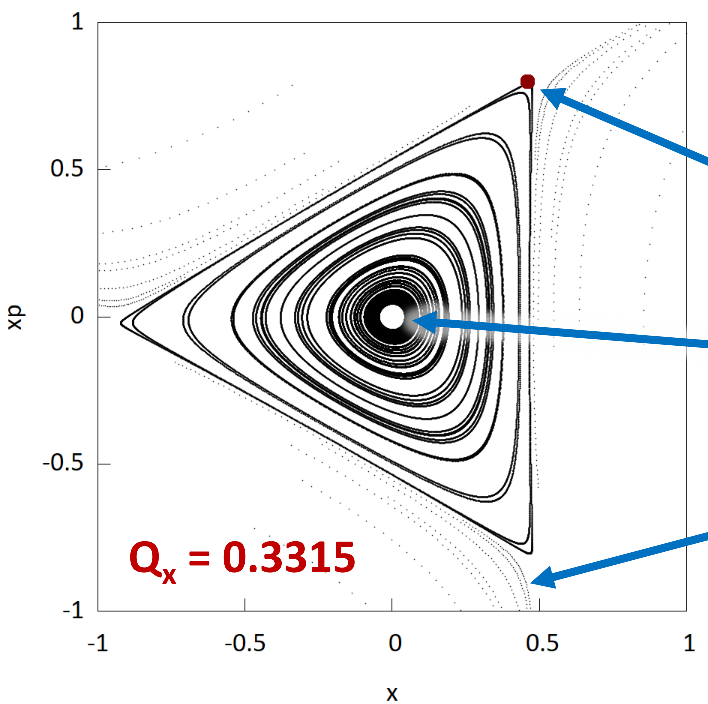
[Poincare section chosen for textbook cover]

Sextupole & Third Order Resonance

step 1: $\begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$

step 2: $\begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} x \\ p_x - \frac{1}{2} k_2 x^2 \end{pmatrix}$ ← sextupole “kick”

repeat this many times for different amplitudes, record coordinates for each turn



fix point, computed from perturbed Hamiltonian

linear motion at small amplitudes

particles close to separatrix, then unbound

3rd Order Resonance in a Nutshell

$$H = \frac{1}{2}p_x^2 + K(s)\,\frac{1}{2}x^2 \quad + \quad S(s)\,x^3$$

linear motion,
harmonic oscillator

sextupole driving
term, distributed

insert undisturbed betatron oscillation

$$x^3 \;\propto\; \cos^3\varphi = \frac{1}{4}(\cos 3\varphi + 3\cos\varphi)$$

third harmonic appears

if the tune Q is close to a third order resonance:

$$\varphi_{n+1} = \varphi_n + 2\pi Q$$
$$= \varphi_n + 2\pi\frac{\mathbf{m}}{\mathbf{3}}$$

3φ varies slowly; distortion adds up coherently

Hamiltonian Formalism

A dynamical system is described by a Hamiltonian with

q_k = coordinates, p_k = canonical momenta, t = independent variable (time).

$$H(q_k, p_k, t)$$

H is often the total energy of a system

The equations of motion: Hamilton's equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \ \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

instead of k second order equations we have $2k$ first order equations

it holds:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_k \underbrace{\left(\frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k \right)}_{0}$$

H = const if not explicitly depending on time

Canonical Transformations of $\mathbf{H}$

H can be transformed/adapted to a specific problem by ***canonical transformations***, i.e. by introducing new variables

$$P_k = P_k(q_k, p_k),\ Q_k = Q_k(q_k, p_k)$$

$$\mathcal{H} = H + \frac{\partial F}{\partial t}$$

using a **generating function *F*** new Hamiltonian Equations are obtained that still fulfill Hamiltons principle.

$$\dot{Q}_k = \frac{\partial \mathcal{H}}{\partial P_k},\ \dot{P}_k = -\frac{\partial \mathcal{H}}{\partial Q_k}$$

depending on the combination of old q, p and new Q, P four different types of F exist (see literature) for example F_3 of old impulses p and new coordinates Q:

$$F = F_3(p, Q, t)$$

$$q = -\frac{\partial F_3}{\partial p},\ P = \frac{\partial F_3}{\partial Q},\ \mathcal{H} = H + \frac{\partial F_3}{\partial t}$$

Hamiltonian – pendulum example

$$H = \frac{1}{2I}p_\theta^2 + \frac{gI}{l}(1-\cos\theta)$$

$$H \approx \frac{1}{2I}p_\theta^2 + \frac{gI}{2l}\theta^2$$ (small angles θ)

$\theta \equiv q$ angle variable

$p_\theta = I\dot{\theta}$ angular momentum

$I = ml^2$ moment of inertia

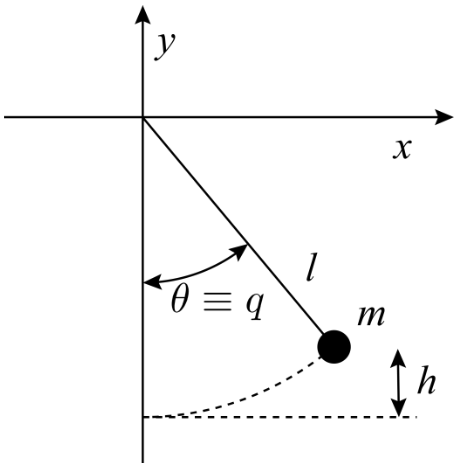
equations of motion :

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{1}{I}p_\theta$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = mgl\sin\theta$$

$$\dot{p}_\theta = I\ddot{\theta}$$

$$\rightarrow \quad \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$



small angles: $\sin\theta \approx \theta \rightarrow$ harmonic oscillator

Pendulum: Action Angle Variables

Harmonic oscillator: $\ddot{\theta} + \frac{g}{l}\theta = 0 \rightarrow \ddot{\theta} + \omega^2\theta = 0, \;\; \omega = \sqrt{\frac{g}{l}}$

Canonical transfomation: $(\theta, p_\theta) \rightarrow (\Psi, J)$

$$H \rightarrow \mathcal{H}$$

Use generating function: $F_1 = -\frac{1}{2}\omega I\theta^2 \tan\Psi$

$$p = \frac{\partial F_1}{\partial x} = -\omega I\theta \tan\Psi$$

$$J = -\frac{\partial F_1}{\partial \Psi} = \frac{1}{2}\omega I\theta^2 \cos^{-2}\Psi$$

$$\mathcal{H} = H + \frac{\partial F_1}{\partial t}$$

↑

0

⇒ now rearrange equations to obtain θ, p_θ as a function of J, ψ

Harmonic Oscillator: Action Angle Variables

Solution of equation of motion in terms of J,ψ:

$$\theta = \sqrt{\frac{2J}{\omega I}}\cos\Psi$$

$$p_\theta = -\sqrt{2J\omega I}\sin\Psi$$

Hamiltonian: $\mathcal{H} = J\cdot\omega$

$\theta \equiv q$ angle variable

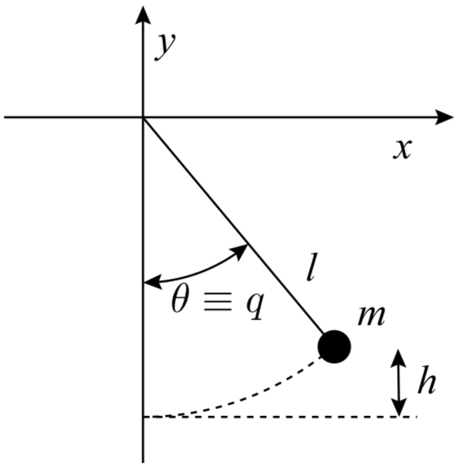
$p_\theta = I\dot{\theta}$ angular momentum

$I = ml^2$ moment of inertia

Thus: $$\dot{\theta} = \frac{\partial\mathcal{H}}{\partial J} = \omega, \quad \dot{J} = -\frac{\partial\mathcal{H}}{\partial\Psi} = 0$$

$$\rightarrow J = \text{const}$$

Canonical Perturbation Theory : Pendulum



moment of inertia: $I = ml^2$

angular momentum: $p_\theta = I\dot{\theta}$

freq. small amplitude: $\omega_0 = \sqrt{g/l}$

Hamiltonian:

$$H = \frac{1}{2I}p^2 + mgl(1 - \cos\theta)$$

$$H = \underbrace{\frac{1}{2I}p^2 + I\frac{\omega_0^2}{2}\Big(\theta^2}_{\text{undisturbed H (harm. oscillator)}} \underbrace{- \frac{1}{12}\theta^4 + \frac{1}{360}\theta^6 - \dots}_{\text{perturbation } \Delta H}\Big)$$

undisturbed H
(harm. oscillator)

perturbation ΔH

action angle variables:

$$\mathcal{H} = \omega_0 J + \Delta\mathcal{H}$$

Canonical Perturbation Theory : Pendulum

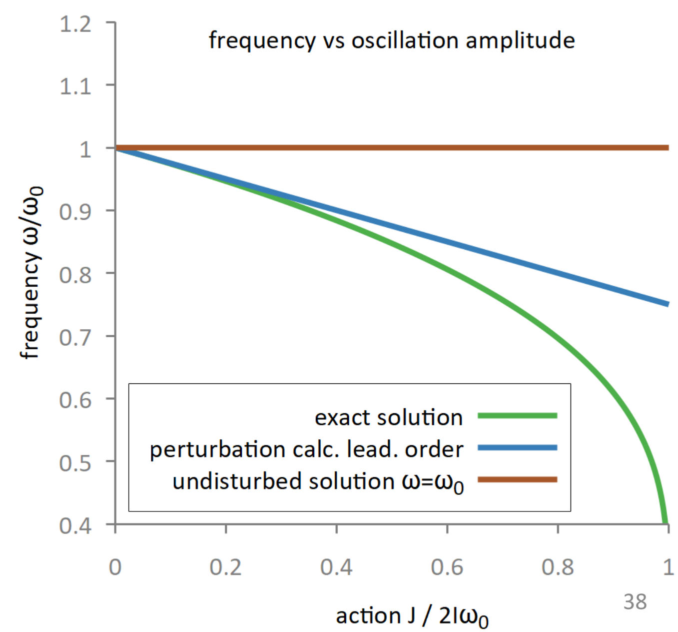
action angle variables (Ψ, J): $$\mathcal{H} = \omega J + \Delta\mathcal{H},\ \Delta\mathcal{H} = -\frac{I}{24}\omega_0^2\,\theta^4$$

insert undisturbed solution: $$\Delta\mathcal{H} = -\frac{1}{6}\frac{J^2}{I}\cos^4(\omega_0 t + \Psi_0)$$

oscillation frequency: $$\Delta\dot{\Psi} = \Delta\omega = \left\langle \frac{\partial \Delta\mathcal{H}}{\partial J} \right\rangle,\ \ \langle \cos^4\Psi \rangle = \frac{3}{8}$$

leading order correction: $$\omega = \omega_0\left(1 - \frac{1}{8}\frac{J}{I}\right)$$

exact solution using elliptic integral K: $$\omega = \omega_0\,\frac{\pi}{2K\left(\sqrt{J/2I\omega_0}\right)}$$



Hamilton Formalism ::

Application to accelerator problems

Hamiltonian for Hill Equation

without proof (see literature on Hamiltonian treatment):

$$\bar{p}_x = p_x/p_0 \approx p_x/p_s = x'$$

$$\bar{p}_y \approx y'$$

$$H = -\frac{\Delta p}{p_0}\frac{x}{\rho} + \left(\frac{1}{\rho^2} + k\right)\frac{x^2}{2} - k\,\frac{y^2}{2} + \frac{1}{2}\bar{p}_x^2 + \frac{1}{2}\bar{p}_y^2$$

applying Hamiltons equations the Hill equations of motion are obtained:

$$x'' + \left(\frac{1}{\rho^2} + k\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0}, \quad y'' - ky = 0$$

*see also Wiedemann sec. 5.4.3

Why use Hamiltonians?

- Hamiltonian equations ensure conservation of phase space
- in curved coordinate systems forces are automatically correct
- equations have same form in every coordinate system
- systematic perturbation treatment of nonlinear forces

Practical Hamiltonians for Accelerators

$$H(x, p_x, s) = K(s)\frac{x^2}{2} + \frac{p_x^2}{2}$$

transformation to action angle variables J, ψ yields:

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H} = \frac{J}{\beta(s)} + \Delta\mathcal{H}$$

Linearizing in s by: $\tilde{\Psi} = \Psi(s) - \int \frac{ds}{\beta_x(s)} + \frac{2\pi Q_x}{C} s$

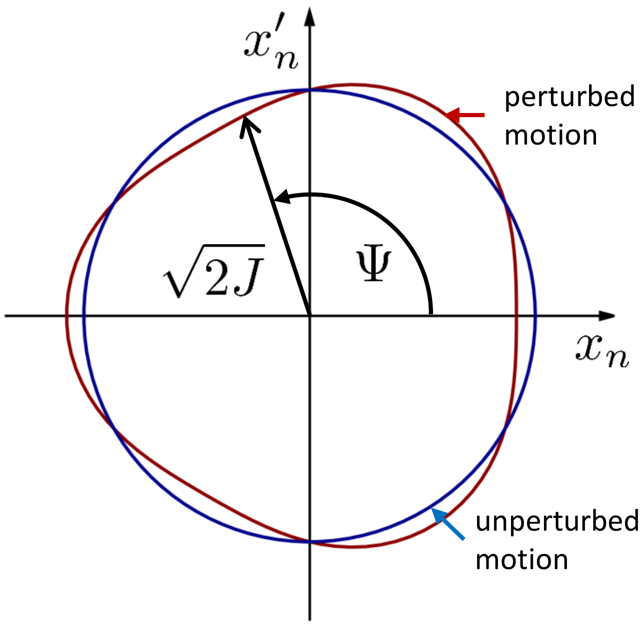
$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}_0} + \Delta\tilde{\mathcal{H}} = JQ_x + \Delta\tilde{\mathcal{H}}$$

Note similarity to pendulum example:

$$\mathcal{H} = \omega J + \Delta\mathcal{H}$$

→ These two are the main types of transformed Hamiltonians used for accelerator problems (we drop ~ for simplicity)

Canonical Perturbation Theory



$$x_n = \sqrt{2J}\cos\Psi$$

$$x'_n = \sqrt{2J}\sin\Psi$$

Hamiltonian type H, linearized motion

unperturbed motion "circle"

small perturbation "nonlinearity"

$$\mathcal{H}(\Psi, J, \theta) = Q_0 J + U(\Psi, J, \theta)$$

betatron phase variable

angle around ring, "time" variable

Fourier Expansion of Perturbation

perturbation is double Fourier expanded w.r.t. phase angle and accelerator azimut:

$$U(\Psi, J, \theta) = \sum_{m,n} U_{m,n}(J) \cdot e^{i(m\Psi - n\theta)}$$

change of angles per revolution:

$$\theta \to \theta + 2\pi$$

$$\Psi \to \Psi + 2\pi \, Q_0$$

if $Q_0 \approx n/m$ this term varies slowly and particles are coherently excited
$\rightarrow$ **"resonance"**
other terms can be neglected

canonical transformation to focus on one resonance:

$$F_2(\Psi_1, J_2, \theta) = \left(\Psi_1 - \frac{n}{m}\theta\right) J_2$$

$$\mathcal{H}_2(\Psi_2, J_2) = \delta J_2 + U_{0,0}(J_2) + 2U_{m,n}(J_2) \cos m\Psi_2 \qquad \delta = Q_0 - \frac{n}{m}$$

Simplified Hamiltonian for particular term

$$\mathcal{H}(\Psi, J) = \delta J + \alpha(J) + A(J)\cos m\Psi$$

driving term

detuning

$$\delta = Q_0 - \frac{n}{m}$$ is small

- multipole fields drive certain resonances in leading order, for example sextupole: 3rd order resonance, octupole: 4th order
- detuning is a shift of the betatron frequency with oscillation amplitude; e.g. octupole $\alpha \propto J^2$
- note: higher order resonances are also present, besides the leading order resonances

Perturbation Treatment of Sextupole

sextpole magnets cause a term:

$$U(\Psi, J, \theta) = S(\theta) \cdot x^3 = S(\theta) \cdot (2\beta J)^{\frac{3}{2}} \cos^3 \Psi$$

$$\frac{1}{4}(\cos 3\Psi + 3\cos\Psi)$$

sextupole strength as a function of θ

Fourier series of U contains terms for m=1, m=3; we focus on the more interesting m=3; there is no detuning term in lowest order

in a ring there are many contributions to $S(\theta)$ which must be summed up with their individual phases; in practice this gives room to minimize the driving term while still correcting chromaticity

Sextupole treatment cont.

Hamiltonian for sextupole:

$$\mathcal{H}(\Psi, J) = \delta J + AJ^{\frac{3}{2}} \cos 3\Psi, \ \delta = Q - \frac{n}{3}$$

Fix-points are points in phase space that do not move over time (here θ):

$$\frac{\partial J}{\partial \theta} = -\frac{\partial \mathcal{H}}{\partial \Psi} = 0 \rightarrow \quad \sin 3\Psi = 0$$

$$\frac{\partial \Psi}{\partial \theta} = \frac{\partial \mathcal{H}}{\partial J} = 0 \rightarrow \quad \delta + \frac{3}{2} AJ^{\frac{1}{2}} \cos 3\Psi = 0$$

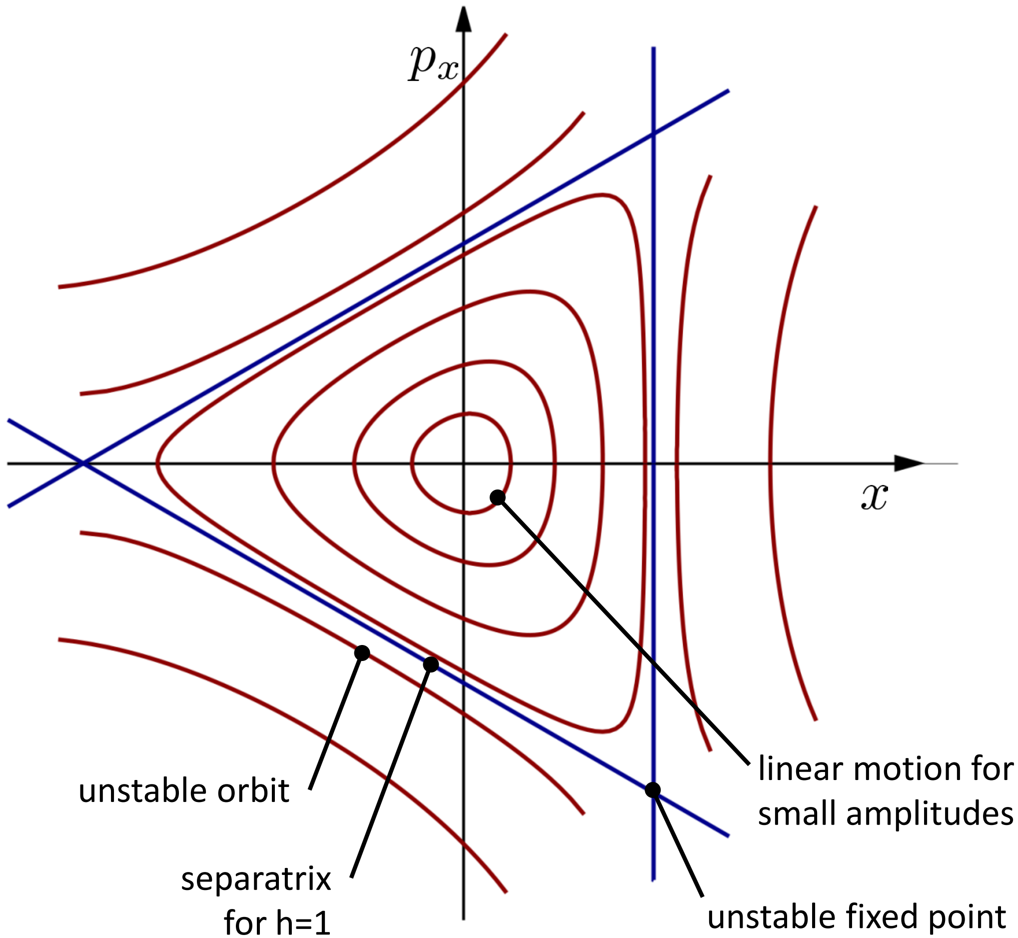
↑

$$\cos 3\Psi = -1 \rightarrow \Psi = \pi/3, \pi, 5\pi/3$$

normalising the Hamiltonian for studies by introducing variables j, h:

$$j = \frac{J}{J_{\mathrm{F.P.}}}, \ J_{\mathrm{F.P.}} = \left(\frac{2\delta}{3A}\right)^2, \quad h = \frac{\mathcal{H}}{\mathcal{H}_{\mathrm{F.P.}}}, \ \mathcal{H}_{\mathrm{F.P.}} = \frac{4}{27}\frac{\delta^3}{A^2}$$

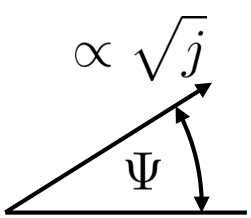
Phase Space Characteristics



this equation describes trajectories in phase space for varying amplitude (h):

$$2j^{\frac{3}{2}}\cos 3\Psi + 3j - h = 0$$

note relation to p_x, x coordinates

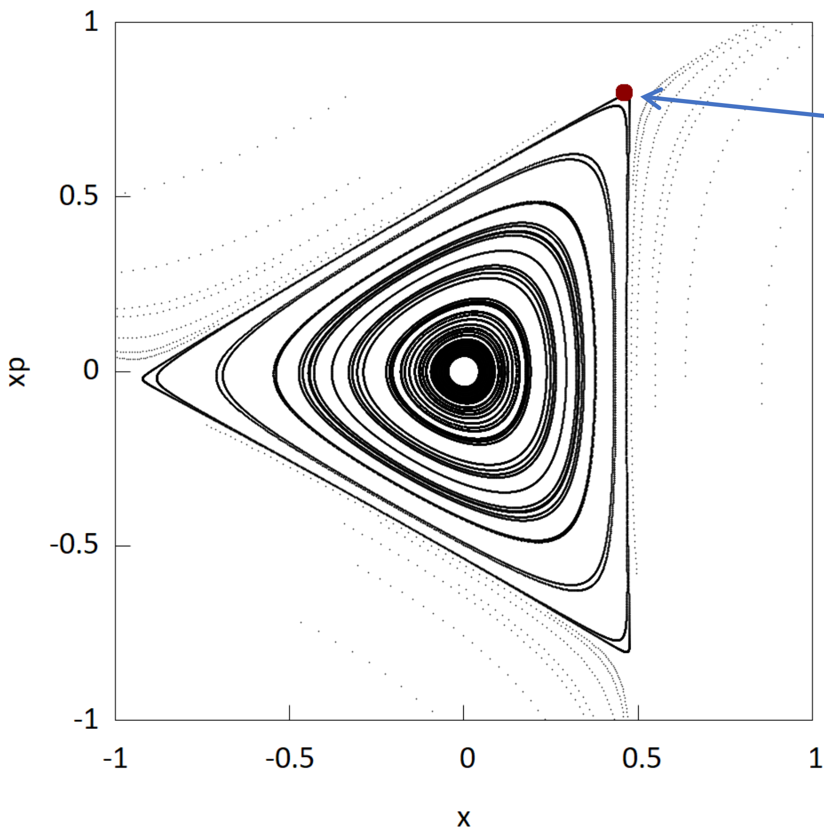


Comparison with tracking simulation

step 1: $$\begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\mu_x) & \sin(\mu_x) \\ -\sin(\mu_x) & \cos(\mu_x) \end{pmatrix} \begin{pmatrix} x \\ p_x \end{pmatrix}$$

step 2: $$\begin{pmatrix} x \\ p_x \end{pmatrix} \rightarrow \begin{pmatrix} x \\ p_x - \frac{1}{2} k_2 x^2 \end{pmatrix}$$

sextupole “kick”



fixed point coordinates computed with perturbation theory

$$J_{\mathrm{F.P.}} = 2\left(\frac{8\pi\delta}{k_2}\right)^2$$

$$\Psi_{\mathrm{F.P.}} = \frac{\pi}{3}$$

Q_x = 0.3315 → δ = -0.0018

application to coupling

Coupling Resonances

coupling of horizontal and vertical betatron motion may be caused by:

skew quadrupole:	$A_s(x,y,s) = f(s)\cdot xy$
normal sextupole:	$f(s)(x^3 - 3xy^2)$
skew sextupole:	$f(s)(3x^2y - y^3)$
normal octupole:	$f(s)(x^4 - 6x^2y^2 + y^4)$
skew octupole:	$f(s)(4x^3y - 4xy^3)$

$\rightarrow$ here we focus on skew quad $\propto xy$ as the simplest and most common effect

Perturbation Treatment of Coupling Resonances

add the skew quad potential in Hamiltonian of type H_4:

$$H = \underbrace{\frac{1}{2}K(s)(y^2 - x^2) + \frac{1}{2}(p_x^2 + p_y^2)}_{\text{unperturbed } H_0} + \underbrace{f(s) \cdot xy}_{\text{perturbation } H_p}$$

then insert the known solutions:

$$x(s) = \sqrt{2J_x\beta(s)}\cos\left(\varphi_x(s) + \varphi_{x0}\right)\,,\;\; y(s) = \ldots$$

use the constants of unperturbed solution as new variables
(known as variation of constants):

$$J_x(s), \varphi_{x0}(s), J_y(s), \varphi_{y0}(s)$$

→ see two slides in appendix with more information on the calculus

Behavior of resonant Hamiltonian

Equations of motion (see appendix):

$$\frac{\partial J_I}{\partial \theta} = \quad \kappa_{q,\pm}\sqrt{J_I J_{II}}\sin\left(\varphi_I \pm \varphi_{II}\right)$$

$$\frac{\partial J_{II}}{\partial \theta} = \pm\kappa_{q,\pm}\sqrt{J_I J_{II}}\sin\left(\varphi_I \pm \varphi_{II}\right)$$

For (+) case : subtract both equations:

$$\frac{\partial}{\partial \theta}(J_I - J_{II}) = 0, \quad \rightarrow J_I - J_{II} = \mathrm{const}$$

In case of the **sum resonance** (+) both betatron amplitudes can grow indefinitely and **particle is unstable**.

Difference Resonance

For (-) case : add both equations:

$$\frac{\partial}{\partial\theta}(J_I + J_{II}) = 0, \quad \rightarrow J_I + J_{II} = \mathrm{const}$$

In case of the difference resonance (-) the sum of the betatron amplitudes is preserved, but action can be exchanged. $\rightarrow$ The **particle is stable**.

Equations of motions can be solved using the perturbation treatment with the ansatz:

$$w = \sqrt{J_I}e^{i\varphi_I}, \quad v = \sqrt{J_{II}}e^{i\varphi_{II}}$$

This leads finally to the solution obtained previously from matrix arithmetic.

$$J_I(\theta) = \frac{A}{4\Omega^2}\left(\Delta^2 + \kappa^2\cos^2(\Omega\theta)\right)$$

$$J_{II}(\theta) = \frac{A}{4\Omega^2}\left(\kappa^2\sin^2(\Omega\theta)\right)$$

$$J_1(0) \neq 0; \ J_2(0) = 0$$

$$\Delta = Q_1 - Q_2 - n$$

$$\Omega = \frac{1}{2}\sqrt{\kappa^2 + \Delta^2}$$

Slide 28, see also Wiedemann, Chapter 20.

Generalised Resonance Condition with Coupling

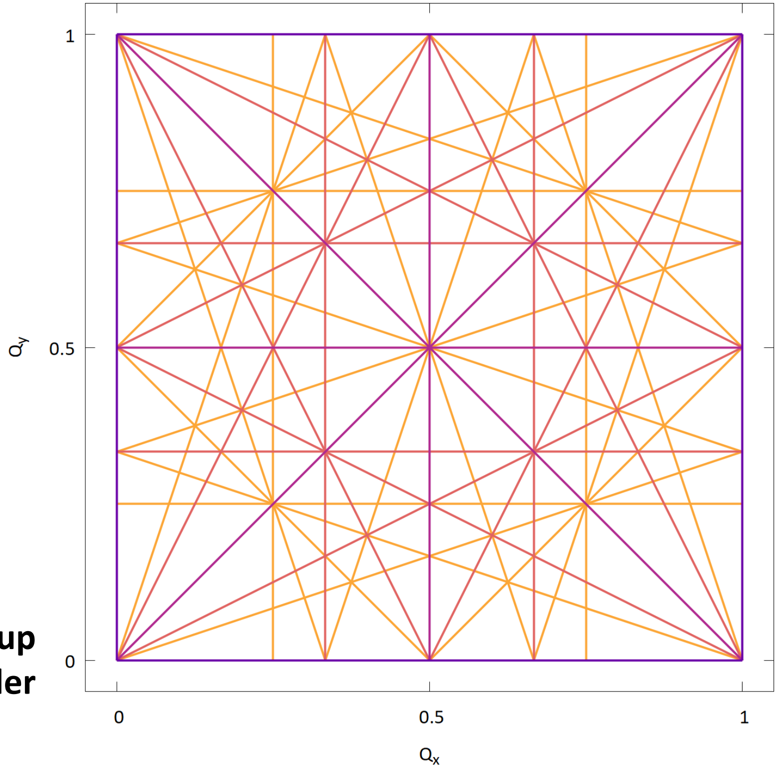
$$U(\Psi_x, \Psi_y, J_x, J_y, \theta) = \sum_{m_1, m_2, n} U_{m_1, m_2, n}(J_x, J_y) \cdot e^{i(m_1 \Psi_x + m_2 \Psi_y - n\theta)}$$

increment of phase term per turn:

$$\Delta(\text{phase}) = 2\pi(m_1 Q_x + m_2 Q_y) - n\theta$$

resonance condition:

$$m_1 Q_x + m_2 Q_y - n \approx 0$$



resonances up to 4th order

What was discussed in Transverse Dynamics III?

- orbit distortions by unwanted dipole errors are corrected using additional corrector magnets, after orbit measurement and (e.g.) SVD matrix inversion
- Quadrupole errors result in tune shifts and beta beating at 2φ
- Coupling: sum resonance $Q_x = -Q_y$ unstable; $Q_x = Q_y$ stable
- Hamilton Formalism allows to treat nonlinear problems in systematic approach
- A transformed Hamiltonian has the form H = $QJ + \Delta$ H, where the effect of ΔH can be treated using perturbation theory
- Fourier expansion of the driving term explains a Zoo of resonance conditions $m_1Q_x+m_2Q_y=n$, that is observed in operating rings

Perturbation Treatment of Coupling II

Perturbation in the new Hamiltonian has the form:

$$\Delta\mathcal{H} = g(s)\sqrt{J_x J_y}\underbrace{\cos\left(\varphi_x + \varphi_{x0}\right)\cos\left(\varphi_y + \varphi_{y0}\right)}$$

from the trigonometric functions **sum and difference terms** of phases are generated, leading to sum and difference resonances:

$$\begin{aligned}\Delta\mathcal{H} &\propto \left(e^{i(\varphi_x+\varphi_{x0})} + e^{-i(\varphi_x+\varphi_{x0})}\right)\left(e^{i(\varphi_y+\varphi_{y0})} + e^{-i(\varphi_y+\varphi_{y0})}\right)\\ &\propto \left(e^{i(\varphi_x+\varphi_y+\dots)} + e^{i(\varphi_x-\varphi_y+\dots)} + \dots\right)\end{aligned}$$

using Fourier expansion to collect all coupling contributions around the ring:

$$\kappa_{q,l} = \frac{1}{2\pi}\int_0^L ds\, g(s)\sqrt{\beta_x\beta_y}e^{i\left(\varphi_{x0}+l\varphi_{y0}-(Q_x+lQ_y-qN)\frac{2\pi}{L}s\right)}$$

Perturbation Treatment of Coupling III

resulting Hamiltonian using independent (eq. time) variable $\theta = s/R$:

$$\Delta\tilde{\mathcal{H}} = \sum_{q,l} \kappa_{q,\pm}\sqrt{J_x J_y}\cos\left(\varphi_{x0} \pm \varphi_{y0} + \Delta_q\theta\right), \quad \Delta_q = Q_x \pm Q_y - qN$$

$\pm$ for sum/diff. resonance

resonance condition, term small

note advancement of different phase variables per turn: $\theta \rightarrow \theta+2\pi$, $\varphi_{x,y} \rightarrow \varphi_{x,y}+2\pi Q_{x,y}$
due to the resonance condition a small distortion will add up over many turns

eliminate explicit dependence on the (time) variable θ with a generating function

$$F(J_I, J_{II}) = J_I\left(\varphi_{x0} - \frac{1}{2}\Delta_q\theta\right) + J_{II}\left(\varphi_{y0} \pm \frac{1}{2}\Delta_q\theta\right)$$

results in "resonant Hamiltonian" H_r ($\varphi_{x0,y0} \rightarrow \varphi_{I,II}$):

$$\mathcal{H}_r = \frac{1}{2}\Delta_r(J_I \pm J_{II}) + \kappa_{q,\pm}\sqrt{J_I J_{II}}\cos\left(\varphi_I \pm \varphi_{II}\right)$$

Hamiltonian Systems: symplectic conditions valid for the 4x4 matrix **M**

Symplecticity 2x2: $$\boldsymbol{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{M}_{x,y}^T \boldsymbol{S} \boldsymbol{M}_{x,y} = \boldsymbol{S}$$

$$\rightarrow 1 \text{ constraint}, \ \det M_x = 1$$

Symplecticity 4x4: $$\boldsymbol{S} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \boldsymbol{M}^T \boldsymbol{S} \boldsymbol{M} = \boldsymbol{S}$$

$$\rightarrow n(2n-1) = 6 \text{ constraints, including } \det M = 1$$

one consequence, the EV's come in reciprocal pairs: $$\lambda_{\mathrm{I}} \cdot \lambda_{-\mathrm{I}} = 1, \ \lambda_{\mathrm{II}} \cdot \lambda_{-\mathrm{II}} = 1$$

In addition since M is real: λ_i and λ_i^* are Eigenvalues