Transverse Dynamics :: Beam Properties and Lattice Design

Laboratory for Particle Accelerator Physics, EPFL

EPFL

Transverse Dynamics - continued

practical questions to be answered:

- ✓ How to ensure bound motion of a particle beam?
- ✓ What are conditions for stability?
- ✓ Amplitude and frequency of particle oscillations?
- Statistical beam properties like beam width and angular spread?
- How to design magnet lattices (arrangements of dipoles and quads in a line)?
- What is the impact of field errors in bending and focusing magnets?

Recap: Stability Criterion – Eigenvalues of **M**

stable for n → ∞ ? $\boldsymbol{M}^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}$

decomposition of M in eigenvectors: $\boldsymbol{M}^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}} = A\lambda_1^n \vec{v}_1 + B\lambda_2^n \vec{v}_2$

complex eigenvalues: $\lambda_1 = e^{-i\mu}, \lambda_2 = e^{i\mu} \rightarrow \lambda_1 + \lambda_2 = 2\cos(\mu) = \mathrm{Tr}\,\boldsymbol{M}$

$$\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \mathrm{Tr}\boldsymbol{M} = a + d$$

motion is stable if $|\mathrm{Tr}\,\mathbf{M}| \leq 2$, which also means that μ is real

Recap: Hills Equation of Motion

$$
\begin{aligned}
x'' + \left(\frac{1}{\rho^2} + k\right) x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' - ky &= 0
\end{aligned}
$$

⇩

$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

DE is valid for

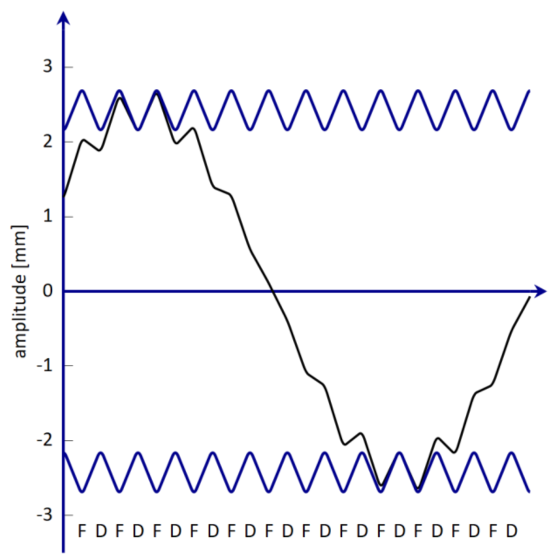
- drift spaces,
- quadrupoles (k≠0),
- combined function magnets (k≠0, 1/ρ≠0),
- off-momentum particles (Δp≠0, first order)

Hill: Solution for periodic K $K(s+C) = K(s)$

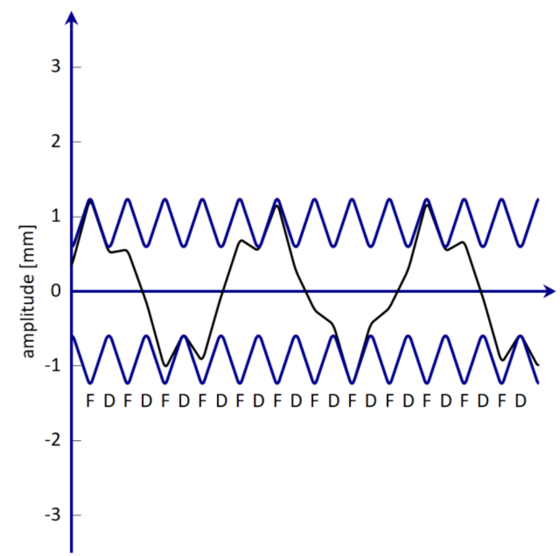
$$x(s) = A\sqrt{\beta(s)}\cos(\varphi(s) - \varphi_0), \;\; \varphi(s) = \int_{t=s_0}^{s} \frac{dt}{\beta(t)}$$

→ the **beta function is a scaling factor** for the amplitude of orbit oscillations and their **local wavelength**

A, φ_0 are constants of motion



weak quads



strong quads

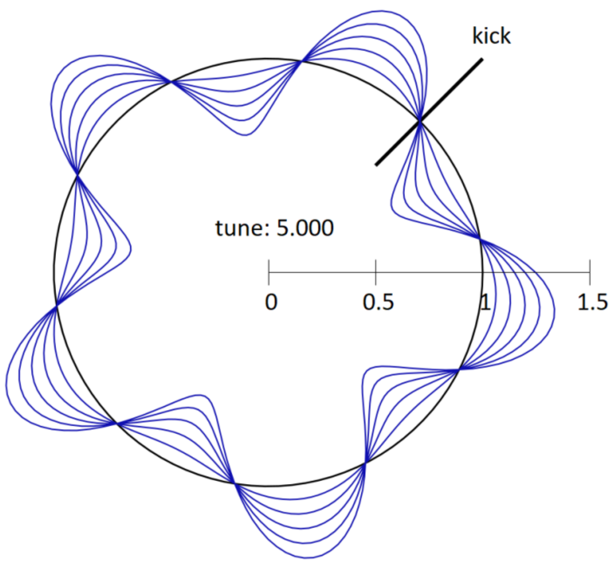
$$A\sqrt{\beta(s)}$$

Recap: The Betatron Frequency Q (tune of accelerator)

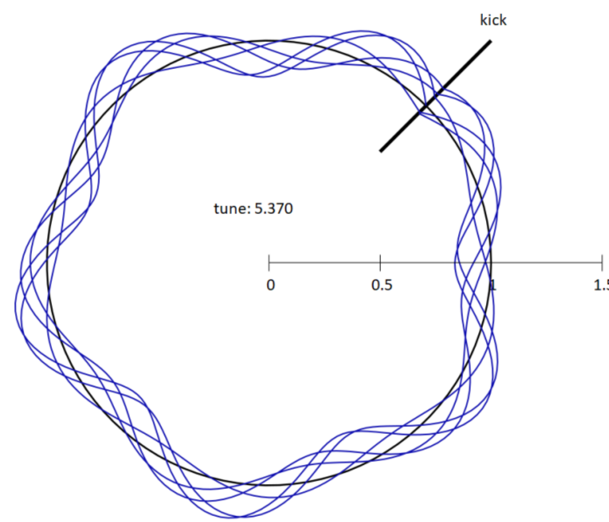
$$Q_x = \frac{1}{2\pi} \oint \frac{ds}{\beta_x(s)}$$

Tune = Number of Betatron Oscillations per Turn
(remember Q=1 for purely weak focusing)

the choice of tune is important to avoid resonances



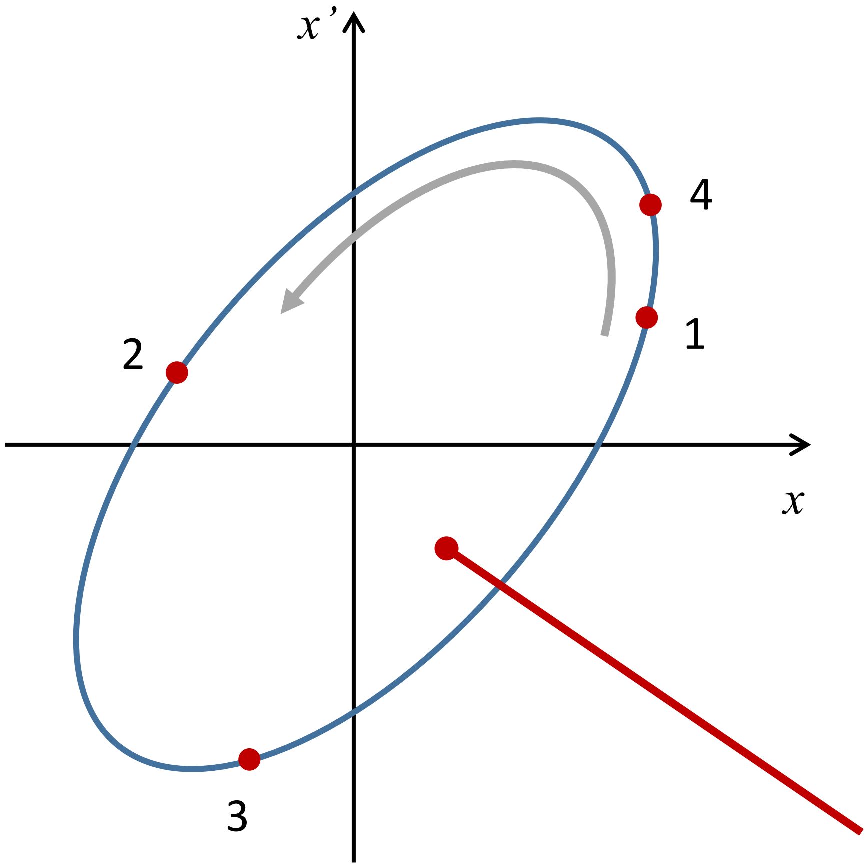
integer tune: resonant growth



odd tune: kick averages out

Phase Space Ellipse

[observing a particle at one location in a ring]



$$x(s) = \sqrt{2J\beta}\cos(\varphi)$$

$$x'(s) = -\sqrt{\frac{2J}{\beta}}\left(\alpha\cos(\varphi) + \sin(\varphi)\right)$$

x, x' describe an ellipse in phase space when φ is varied

J = particle action (oscillation amplitude)

$$\text{area} = 2\pi J = \pi(\gamma x^2 + 2\alpha x x' + \beta x'^2)$$

Next: Statistical Beam Properties

- Liouville theorem
- emittance and distribution function
- consequences of conservation of emittance

Liouvilles Theorem

the phase space density is conserved $\frac{d\psi}{dt} = 0$

continuity equation of 6-dim phase space density ψ:

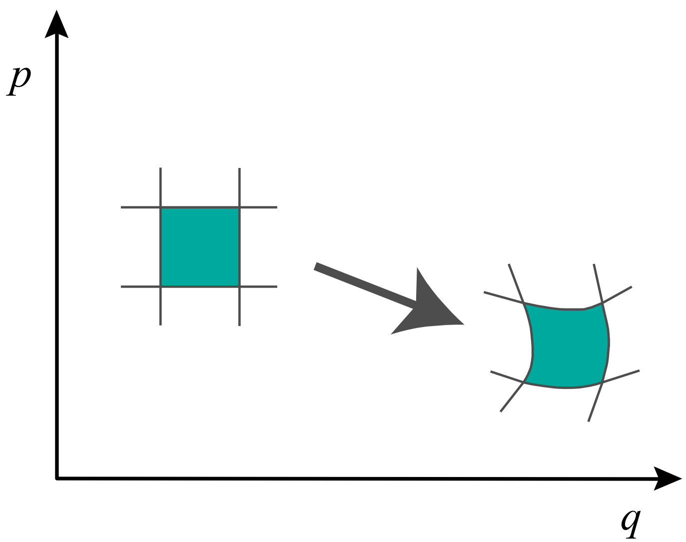
$$\frac{d\psi}{dt} + \vec{\nabla}_6 \vec{j} = 0$$

re-formulated:

$$\frac{d\psi}{dt} + \frac{\partial}{\partial q_k}\dot{q}_k + \frac{\partial}{\partial p_k}\dot{p}_k = 0$$

$$\frac{d\psi}{dt} + \underbrace{\frac{\partial}{\partial q_k}\frac{\partial \mathcal{H}}{\partial p_k} - \frac{\partial}{\partial p_k}\frac{\partial \mathcal{H}}{\partial q_k}}_{= 0 \text{ for Hamiltonian Systems}} = 0$$

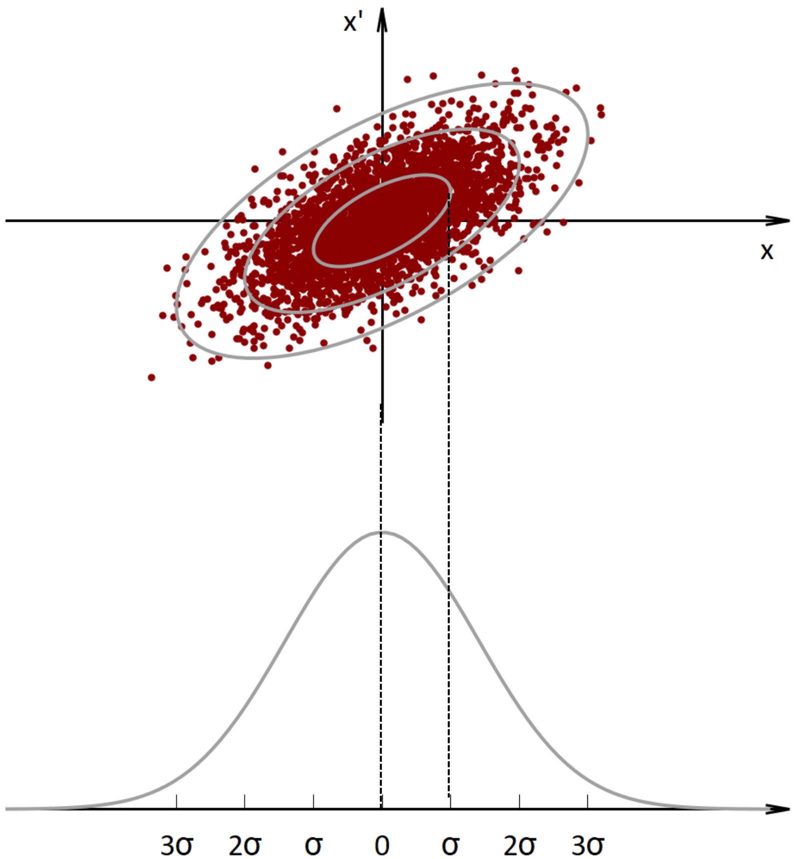
= 0 for Hamiltonian Systems



The phase space density behaves like an incompressible liquid.

see also Wiedemann sec. 8.1.1

Beam Emittance



beam emittance as statistical property:

$$\varepsilon_x = \sqrt{< x^2 >< x'^2 > - < xx' >^2}$$

two-dimensional Gaussian distribution:

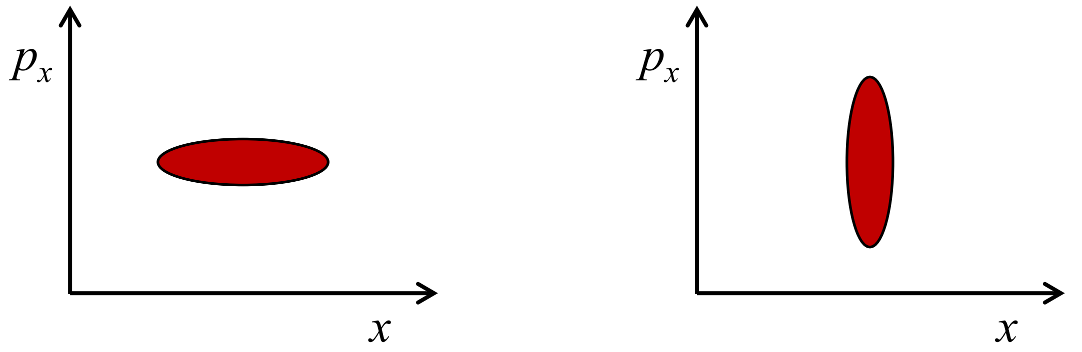
$$f(x, x') = \frac{1}{2\pi\varepsilon_x} \exp\left(-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon_x}\right)$$

projected Gaussian distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sqrt{\beta_x \varepsilon_x}} \exp\left(-\frac{x^2}{2\beta_x \varepsilon_x}\right)$$

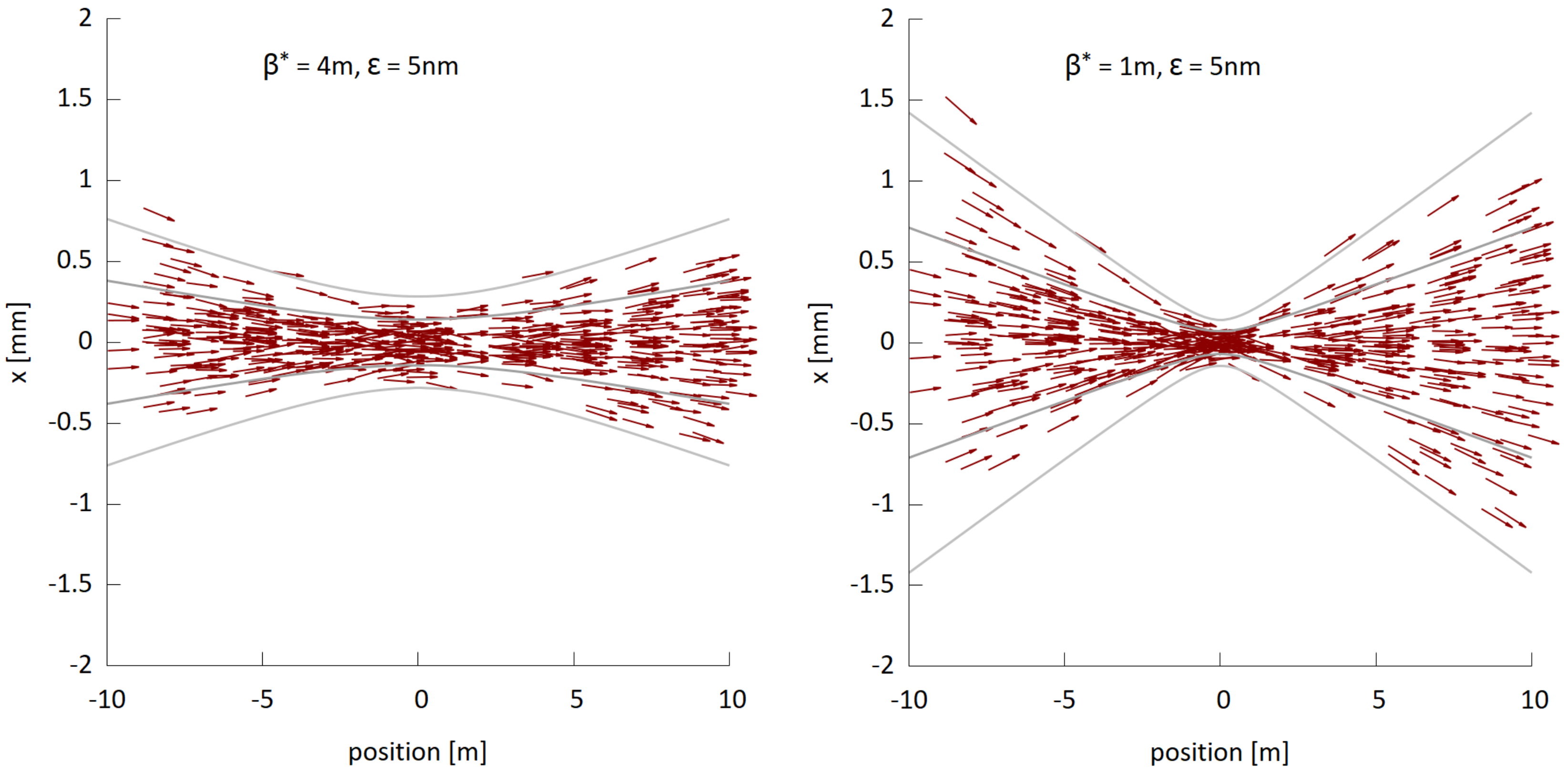
see Wiedemann sec. 8.1.3

Conservation of Emittance



with a given emittance a beam can be made small with large angular spread, or can have small angular spread with a large size

Beam Waist (e.g. interaction point collider)

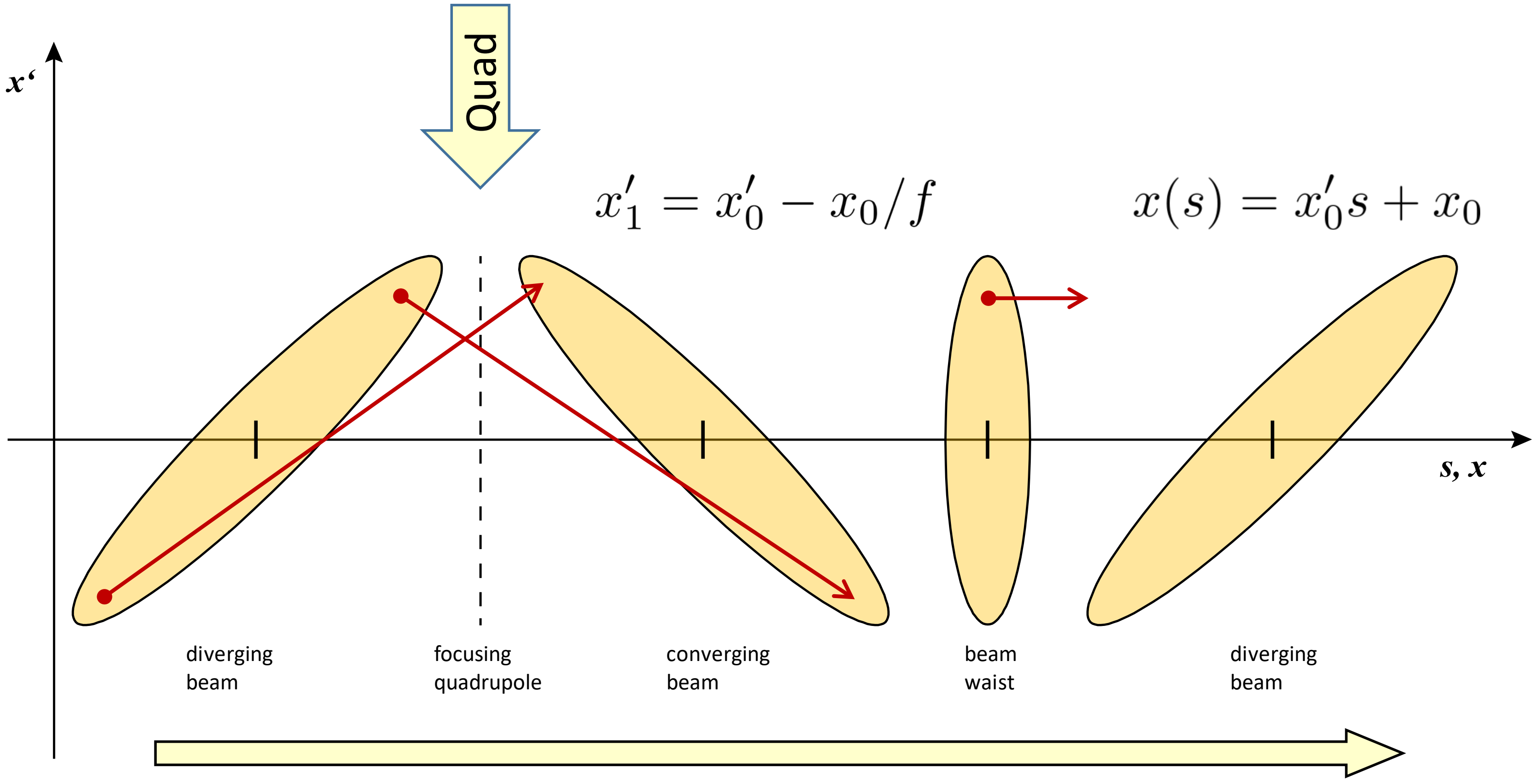


$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

β* = Beta function at waist

$$\sigma_{\mathrm{rms}} = \sqrt{\varepsilon\beta^*},\ \sigma'_{\mathrm{rms}} = \sqrt{\frac{\varepsilon}{\beta^*}} \longrightarrow \sigma_{\mathrm{rms}}\sigma'_{\mathrm{rms}} = \varepsilon = \mathrm{const}$$

Phase Space Ellipse after focusing



Emittance and Twiss Matrix

beam moments are computed in a compact way using the Twiss matrix:

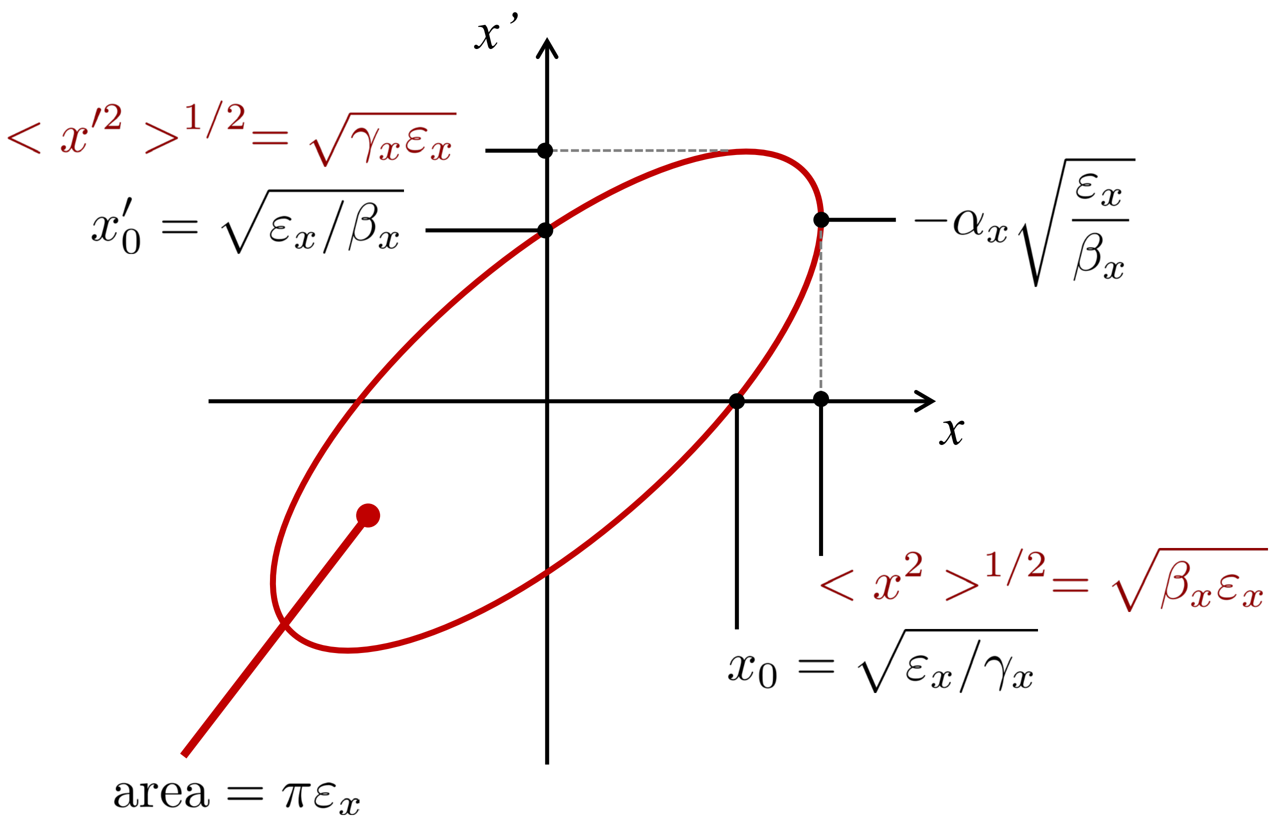
$$\Sigma_x = \varepsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} = \begin{pmatrix} < x^2 > & < xx' > \\ < xx' > & < x'^2 > \end{pmatrix}$$

if Σ_x is known / has been measured, the emittance is related as follows:

$$\varepsilon_x = \sqrt{\det \Sigma_x}$$

see Wiedemann sec. 8.1.3

Phase Space Ellipse - Parameters



reminder:

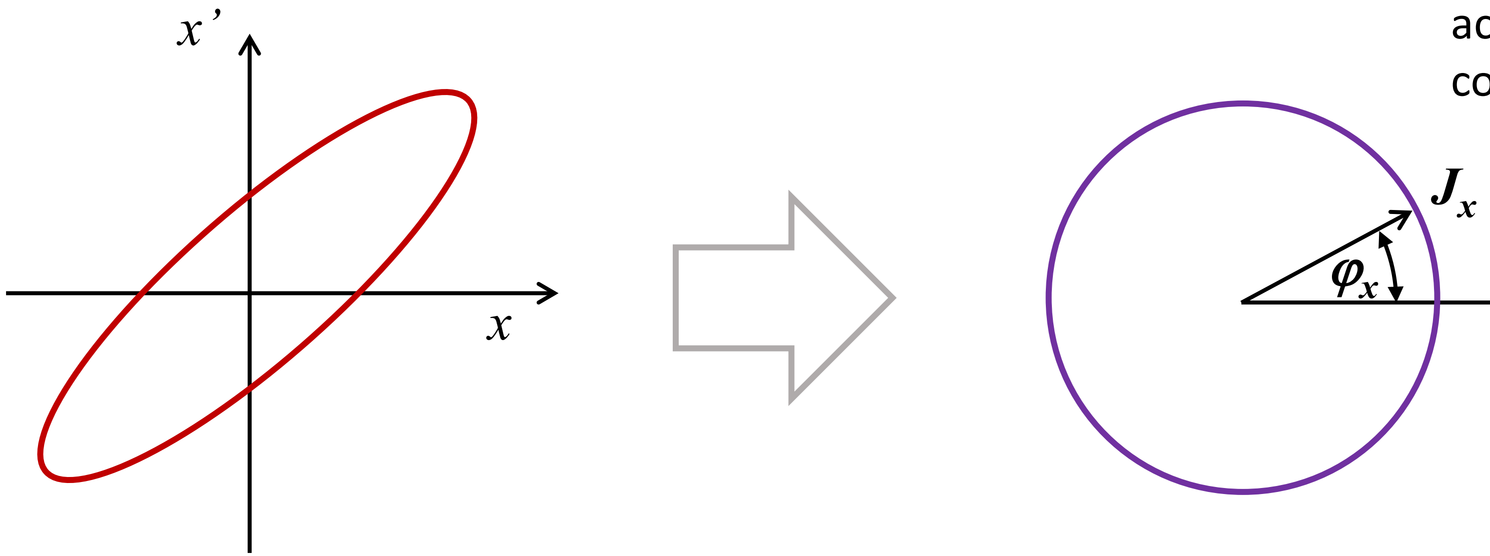
$$\gamma_x = \frac{1 + \alpha_x^2}{\beta_x}$$

$$\alpha_x = -\frac{1}{2}\beta'_x$$

for upright ellipse:

$$< xx' >= 0, \;\; \alpha_x = 0$$

Action – Angle Variables J_x, φ_x



action J_x is conserved

forward transformation:

$$2J_x = \beta\, x'^2 + 2\alpha x x' + \gamma x^2, \;\; \tan\varphi_x = -\alpha - \beta\frac{x'}{x}$$

backward transformation:

$$x = \sqrt{2J_x\beta}\cos(\varphi_x), \;\; x' = -\sqrt{\frac{2J_x}{\beta}}\,(\alpha\cos(\varphi_x) + \sin(\varphi_x))$$

Distribution in Action-Angle Variables

switch to action angle variables:

$$(x, x') \longrightarrow (J_x, \varphi_x)$$

from Gaussian distribution:

$$\rho(x, x')\, dx\, dx' = \frac{N}{2\pi\varepsilon_x}\ \exp\left(-\frac{\beta\, x'^2 + 2\alpha x x' + \gamma x^2}{2\,\varepsilon_x}\right) dx\, dx'$$

to Exponential distribution:

$$\rho(J_x, \varphi_x)\ dJ_x\ d\varphi_x = \frac{N}{2\pi\varepsilon_x} \exp\left(-\frac{J_x}{\varepsilon_x}\right)\ dJ_x\, d\varphi_x$$

the emittance is the average value of the action J_x:

$$< J_x > = \int J_x \rho(J_x, \varphi_x)\ dJ_x d\varphi_x$$

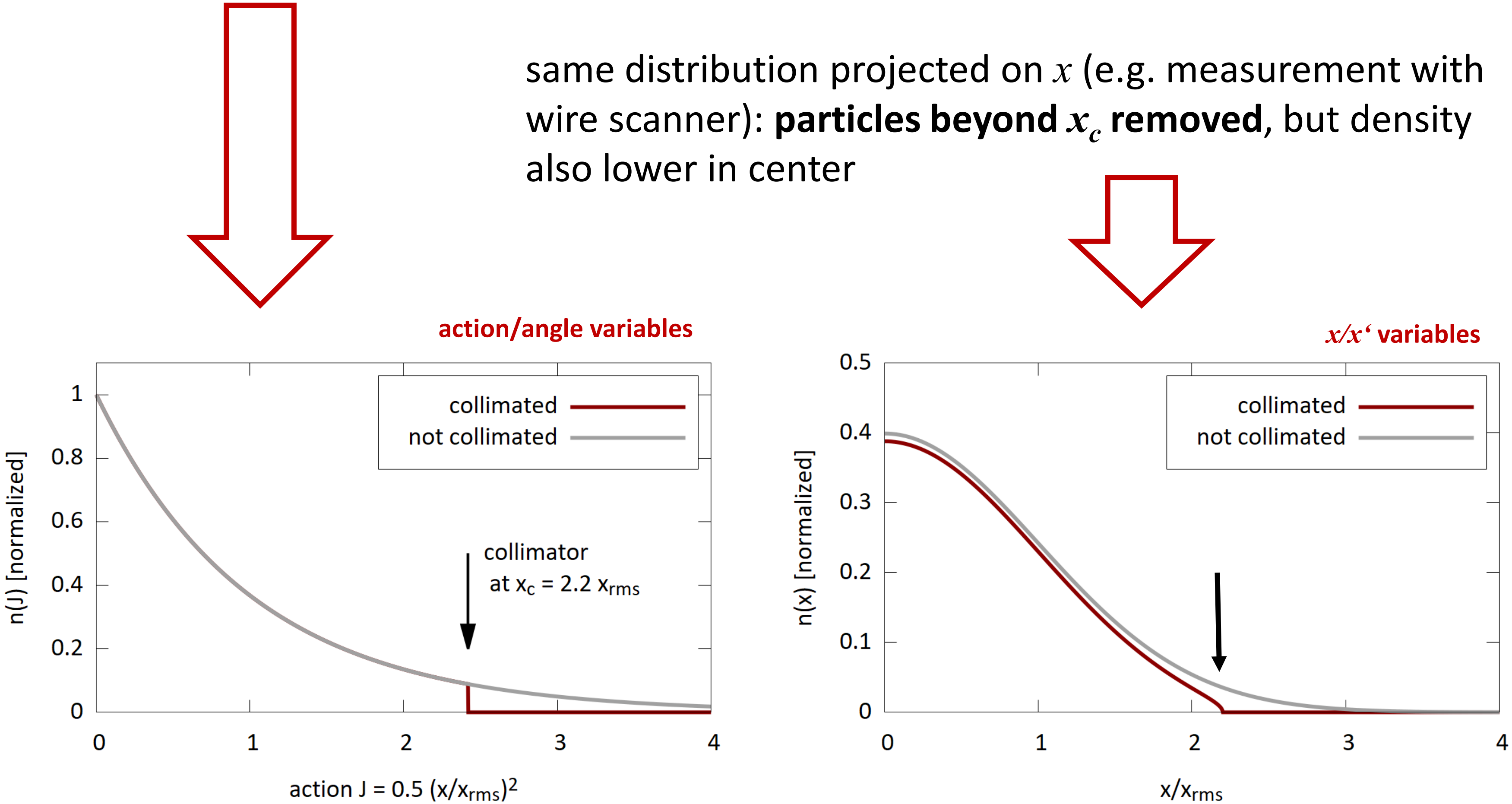
$$< J_x > = \varepsilon_x$$

see Wiedemann sec. 5.4.5

Collimated Proton Beam in x and J_x

circulating beam is collimated, **particles beyond J_c removed**, others not affected

same distribution projected on x (e.g. measurement with wire scanner): **particles beyond x_c removed**, but density also lower in center



Fractions of Beam in rms widths

From practical measurements (wire scan, beam screen) the projected rms width of the beam is determined.
What fraction of beam is contained in $n \times \sigma_{\rm rms}$?

compute the beam fraction inside an ellipse corresponding to n$\times\sigma_x$:

$$J(x = n\sigma_x) = \frac{n^2}{2}\varepsilon_x$$

$$r = \frac{1}{2\pi\varepsilon_x}\int_0^{\frac{n^2}{2}\varepsilon_x}\int_0^{2\pi}\exp\left(-\frac{J}{\varepsilon_x}\right)\,dJ\,d\varphi$$

note: This applies for a two dimensional Gaussian distribution.

$$r = 1 - \exp(-n^2/2)$$

rms width n	beam fraction r
1	39%
2	86%
3	99%

Remarks on Beam Distributions

Electrons

in a ring electrons radiate photons which continuously mixes particles in phase space and generates an equilibrium Gaussian distribution

i.e. a large injected beam will shrink to equilibrium while a small beam will grow

Protons, Ions

“protons never forget” G.Voss

can have “strange” distributions since those depend on the history of beam generation and acceleration; i.e. no damping mechanism

however: in practice often close to Gaussian distribution

Next: FODO Lattices

- FODO parameter space
- FODO with bending magnets

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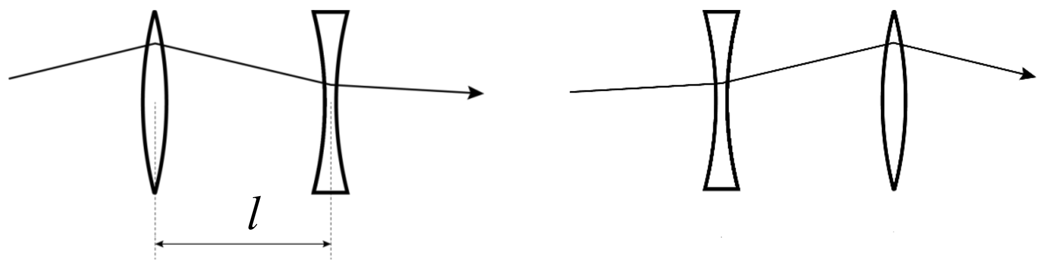
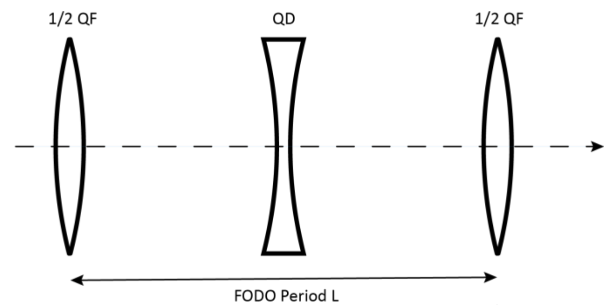
Reminder: Quadrupole Doublet



$$\boldsymbol{M}_{\mathrm{doublet}} = \begin{pmatrix} 1+\frac{l}{f} & l \\ -\frac{1}{f^*} & 1-\frac{l}{f} \end{pmatrix}$$

$$f^* = \frac{f^2}{l} > 0$$ $\rightarrow \mathrm{M}_{\mathrm{doublet}}$ is always focusing

FODO Cell



$$\boldsymbol{M}_{\mathrm{FODO}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) \\ -\frac{1}{f^*} & 1 - \frac{L^2}{8f^2} \end{pmatrix}, \quad \frac{1}{f^*} = \frac{L}{4f^2}\left(1 - \frac{L}{4f}\right)$$

to determine β we use the equation from last lecture and set $\beta_s=\beta_0$, $\alpha_s=\alpha_0=0$

$$\begin{pmatrix} \beta \\ 0 \\ 1/\beta \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC' + S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ 0 \\ 1/\beta \end{pmatrix}$$

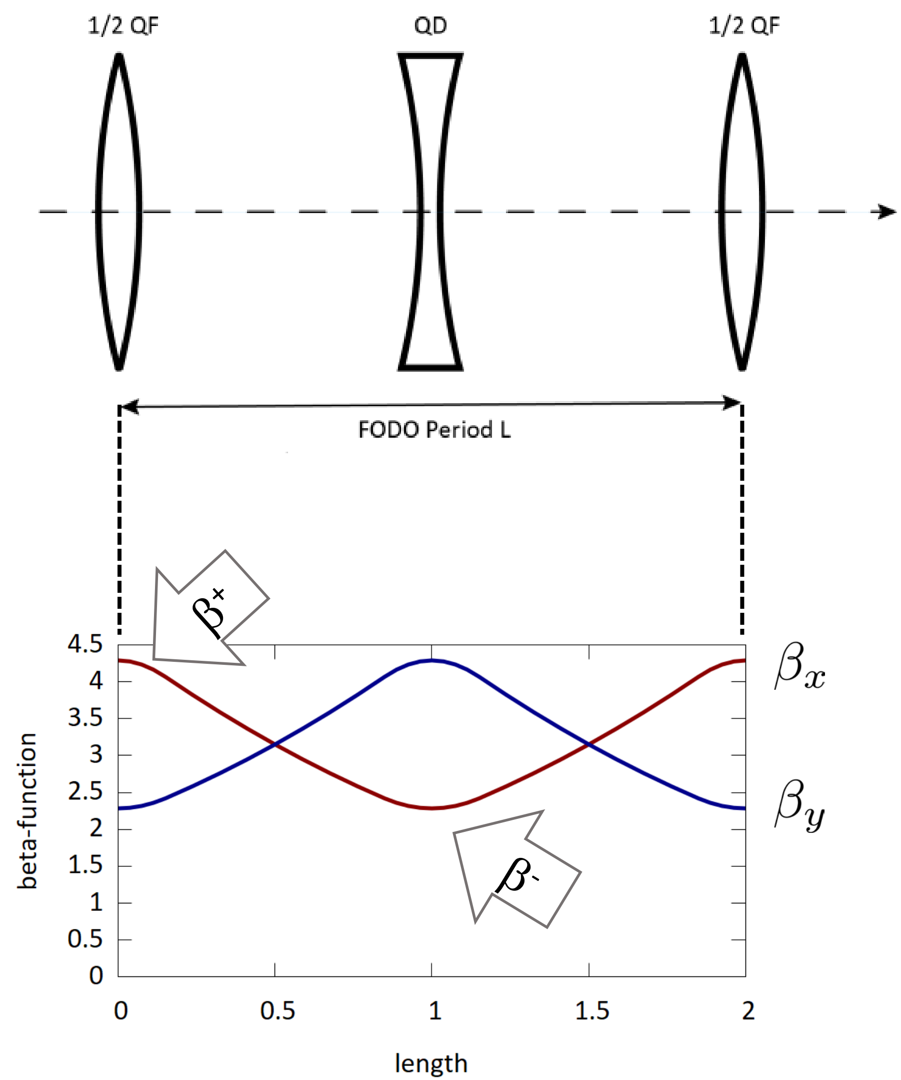
FODO Cell Parameters

we obtain for β^+ in the focusing quad and β^- in the defocusing:

$$\beta^{\pm} = L \, \frac{1 \pm \sin(\mu/2)}{\sin \mu}$$

phase advance per cell:

$$\sin(\mu/2) = \frac{L}{4f}$$



see Wiedemann sec. 10.1

FODO Cell II

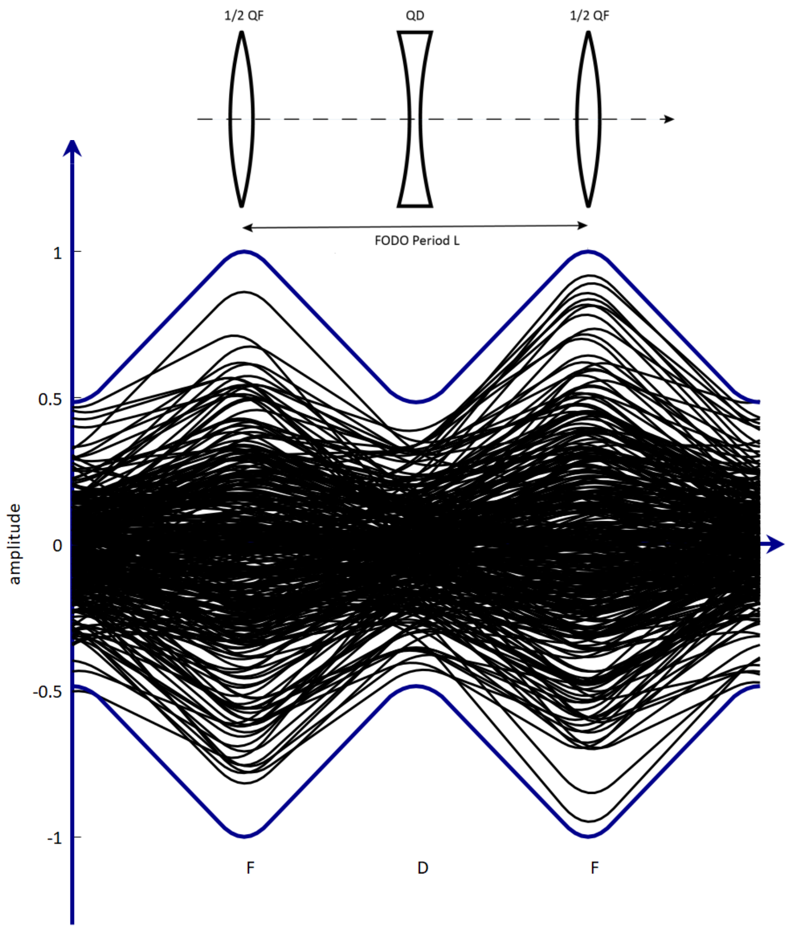


illustration:
particle trajectories of varying phase and amplitude in a FODO cell

Gaussian (projected) profile

FODO Cell: choice of phase advance

β^+ reaches minimum at μ_{opt}=76.3deg
at this point the vacuum chamber
needs a minimal size

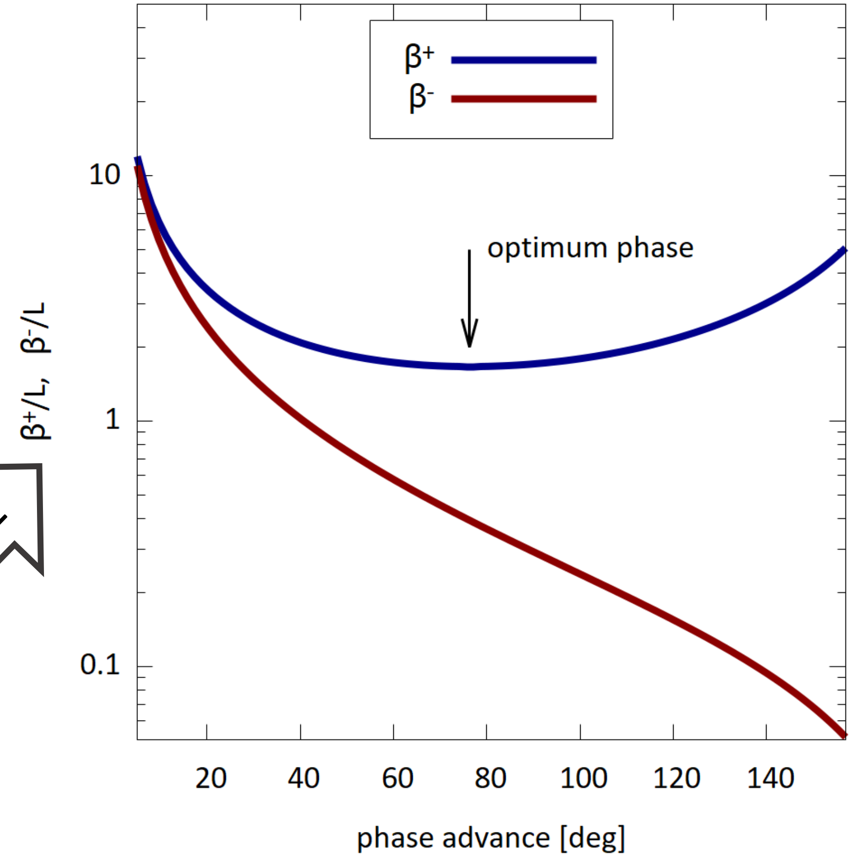
$$\sin(\mu_{\rm opt}) = \frac{2}{1+\sqrt{5}} = \frac{1}{r_g}$$

Example LEP(CERN)
operating modes:

μ_x/μ_y [degrees]:
60/60, 90/60, 90/90, 102/90

(stronger foc. = smaller emittance)

note scaling per L



Unequal F and D quadrupole strength

for uneven strength of the two quads we can compute a region of stability:

convenient variables (dimensionless):

$$F = \frac{L}{4f_F} > 0, \ D = \frac{L}{4f_D} > 0$$

from Trace M condition:

$$\cos\mu = 1 + 2D - 2F - 2DF$$

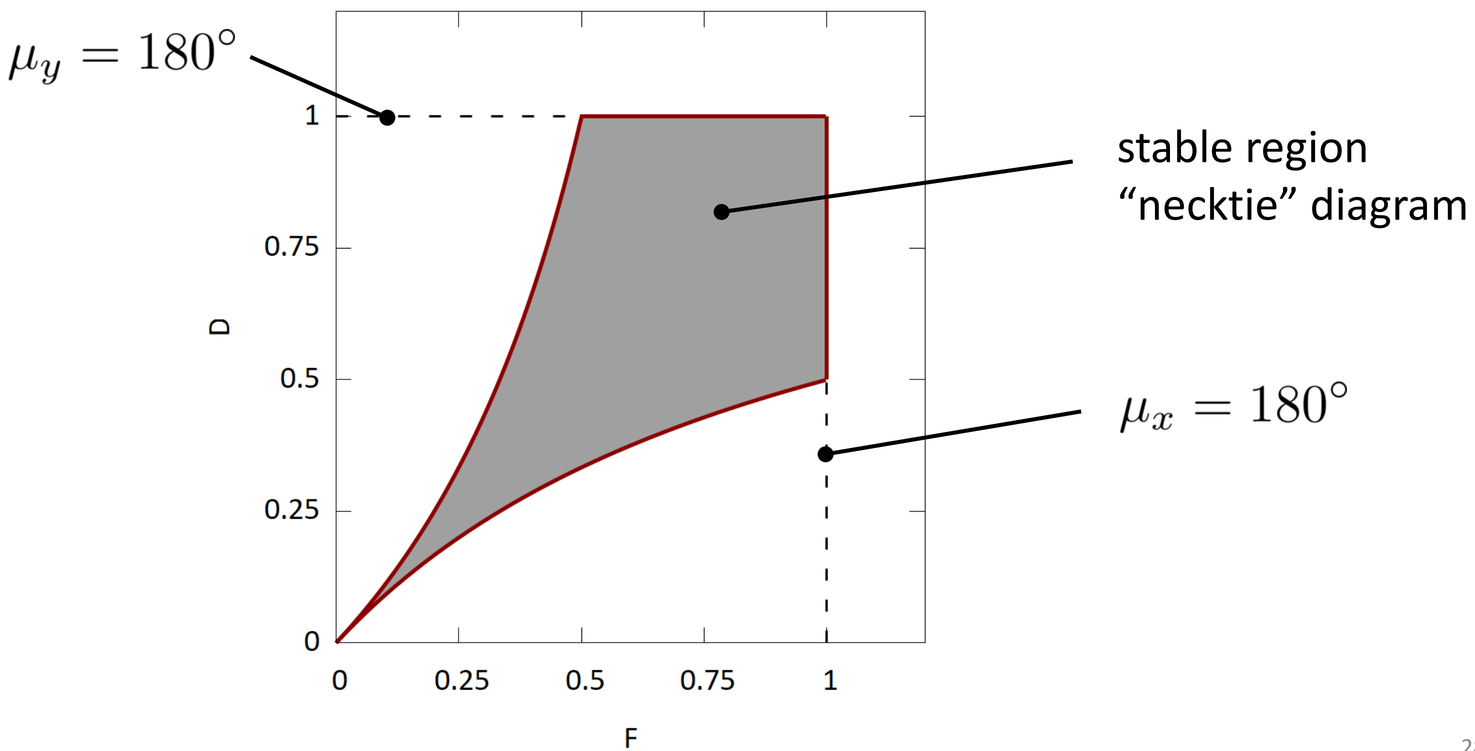
$$\sin^2\mu/2 = DF + F - D = 0\ldots 1$$

it follows this range of focusing strength for stable conditions:

$$\begin{aligned} \text{horizontal} : 0 \leq F - D + FD \leq 1 \\ \text{vertical} : 0 \leq D - F + FD \leq 1 \end{aligned}$$

Stable region for quad strength in a FODO cell

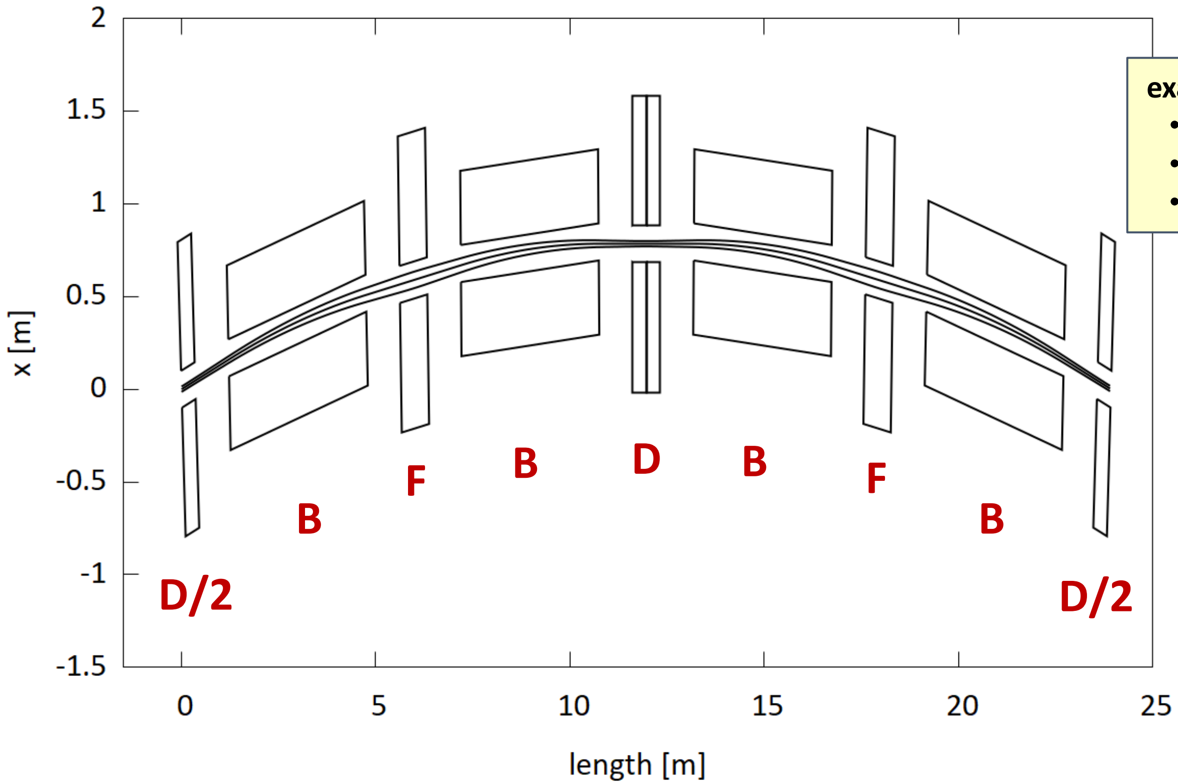
$$\text{horizontal} : 0 \le F - D + FD \le 1$$
$$\text{vertical} : 0 \le D - F + FD \le 1$$



Next: include bending magnets and off-momentum particles into FODO

FODO Cell with Bending Magnets

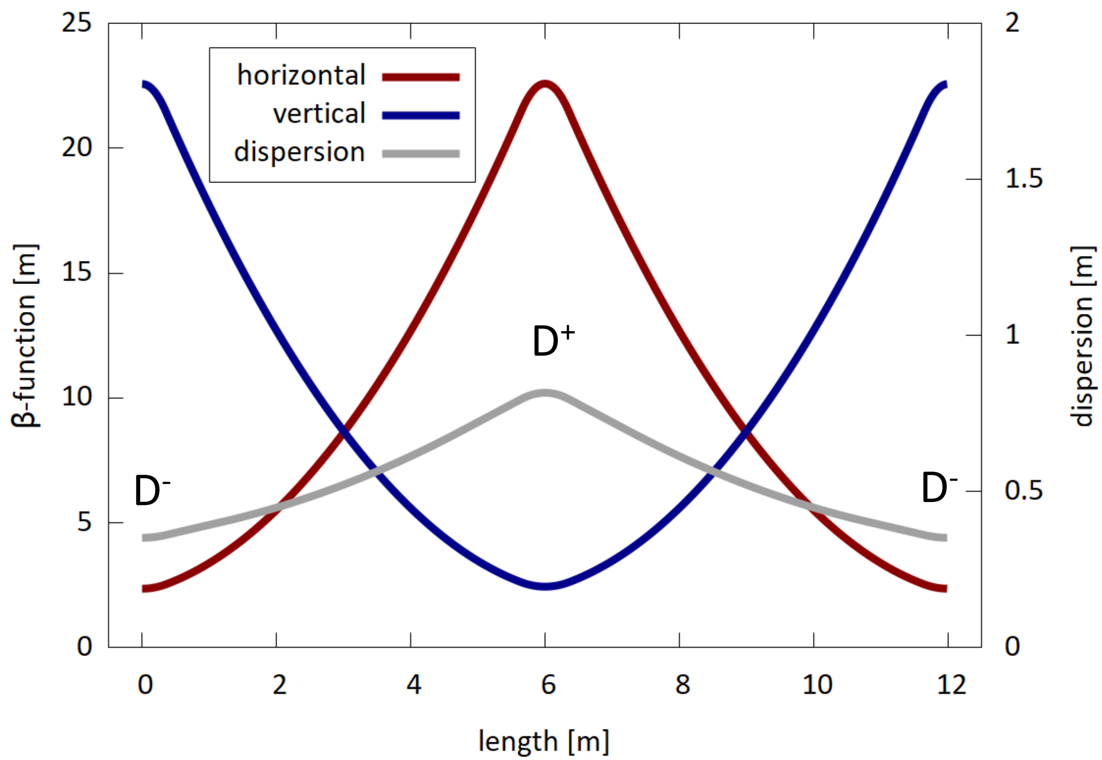
FODO structure with bending magnets to form a ring
- the standard scheme for synchrotrons



example:
- **2 FODO cells**
- **ρ = 55m**
- **separated function**

FODO Cell with Dispersion

dispersion function D(s) is a periodic function in FODO cells with a maximum D^+ in a focusing quad and a minimum D^- in a defocusing quad



see Wiedemann sec. 10.2.3

FODO Cell with Dispersion

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s_0}$$

using the previously introduced 3x3 matrix for transport through ½ FODO cell in thin lens approximation we obtain two equations that are solved for D^+ and D^-

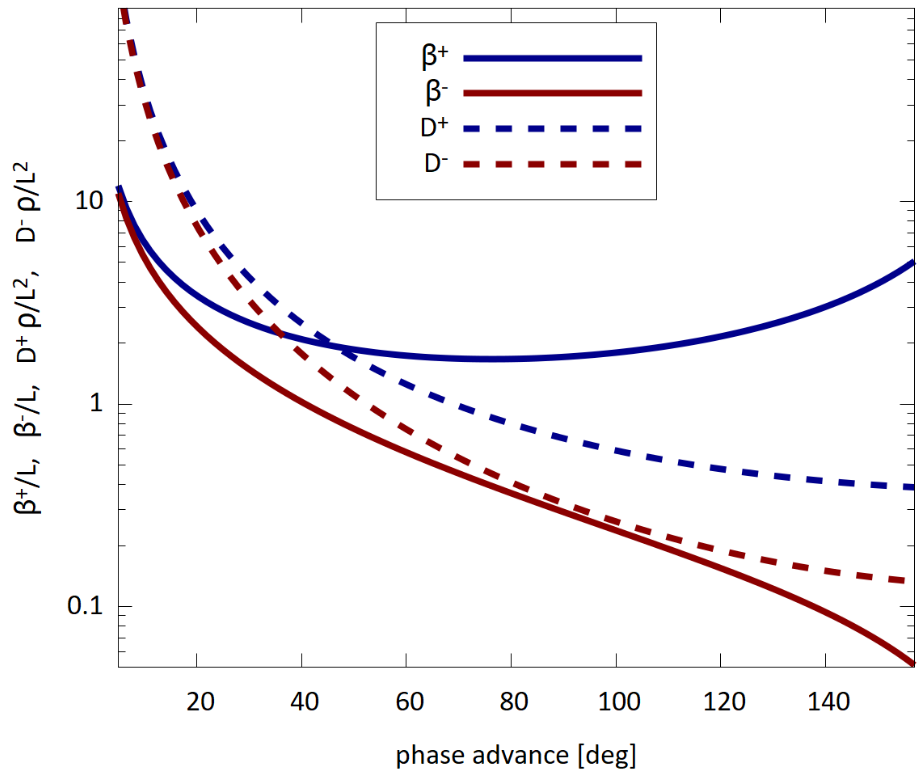
$$\begin{pmatrix} C_1 & S_1 & D^- \\ C_1' & S_1' & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}_{\text{half cell}} \cdot \begin{pmatrix} C_0 & S_0 & D^+ \\ C_0' & S_0' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D^+ = \frac{\theta_c L}{4} \frac{1 + \frac{1}{2}\sin(\frac{\mu}{2})}{\sin^2(\frac{\mu}{2})}, \quad D^- = \frac{\theta_c L}{4} \frac{1 - \frac{1}{2}\sin(\frac{\mu}{2})}{\sin^2(\frac{\mu}{2})}$$

$\theta_c \approx L/\rho$ deflection angle per cell

→ note the quadratic (strong) dependence of D on cell length L

Dispersion Functions vs Phase Advance



note the vertical scaling in graph:

$$D\frac{\rho}{L^2},\ \ \beta\frac{1}{L}$$

→ dispersion depends quadratically on the cell length L
→ with stronger focusing (stronger quads, larger phase advance) the dispersion function gets smaller for the same bending radius

Dispersion Function in a Ring

the dispersion function at position s is calculated by integrating over contributions from bending magnets ($1/\rho \neq 0$) around the ring:

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint dt \, \frac{\sqrt{\beta(t)}}{\rho(t)} \cos\left(\varphi(t) - \varphi(s) - \pi Q\right)$$

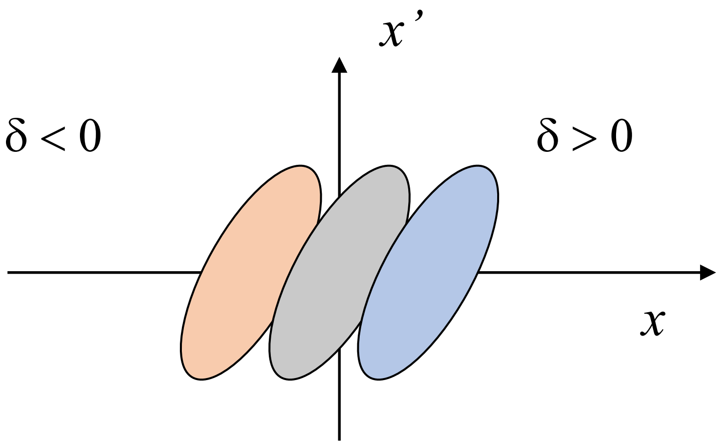
$$D'(s) = \frac{1}{2\sqrt{\beta(s)}\sin(\pi Q)} \oint dt \, \frac{\sqrt{\beta(t)}}{\rho(t)} \left(\alpha(s)\cos(\dots) + \sin(\dots)\right)$$

D, D’ are periodic functions:

$$D(s+C) = D(s), \;\; D'(s+C) = D'(s)$$

Beam size with finite momentum spread

besides emittance also momentum spread may contribute to beam size and angular spread, via dispersion function; when the beam momentum spread is d:



quadratic addition of transverse and longitudinal contributions:

$$\sigma_{\text{tot}}^2 = \sigma_\varepsilon^2 + \sigma_\delta^2 = \varepsilon\beta + D^2 \frac{\delta p^2}{p_0^2}$$

$$\sigma_{\text{tot}}'^{\,2} = \sigma_\varepsilon'^{\,2} + \sigma_\delta'^{\,2} = \varepsilon \frac{1+\alpha^2}{\beta} + D'^2 \frac{\delta p^2}{p_0^2}$$

at some locations the momentum contribution should be suppressed by designing for D=0, D‘=0

examples:

- interaction point in a collider where beams should be as small as possible
- undulators/source magnets, where divergence of emitted radiation should be small

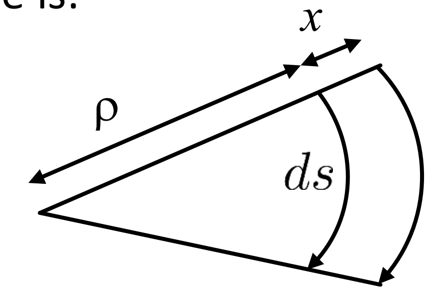
Path Length Change with Momentum

for an off-momentum particle the path length changes; consider a particle on a closed dispersion trajectory, i.e. no betatron oscillation: J=0, but $\Delta p \neq 0$

$$x_D(s) = D(s)\frac{\Delta p}{p_0}$$

the change in circumference for this particle is:

$$\Delta C = \oint \frac{x_D(s)}{\rho(s)} ds$$



$$ds' = ds\left(1 + \frac{x_D}{\rho}\right)$$

we introduce the **momentum compaction factor** α_c:

$$\frac{\Delta C}{C} = \alpha_c \frac{\Delta p}{p}, \;\; \alpha_c = \frac{1}{C}\oint \frac{D(s)}{\rho(s)} ds$$

see Wiedemann sec. 8.4.6

Smooth Approximation (reminder last lecture)

simplify: $$\beta_{\text{avg}} = \langle\beta(s)\rangle = \text{const}$$

$$x(s) \approx A\sqrt{\beta_{\text{avg}}}\cos\left(\frac{s}{\beta_{\text{avg}}} - \varphi_0\right), \;\; x'' + K_{\text{eff}}x = 0$$

can be used to estimate important parameters:

$$K_{\text{eff}} = \frac{1}{\beta_{\text{avg}}^2}$$

$$Q = \frac{1}{2\pi}\oint\frac{ds}{\beta_{\text{avg}}} = \frac{R}{\beta_{\text{avg}}}$$

note: Q $\propto$ R, i.e. proportional to size
compare cyclotron: Q $\propto \gamma$, independent of size!

Smooth Approximation – Dispersion, M.Compaction

$$D'' + K(s)D = \frac{1}{\rho}$$

simplifying assumptions:

$$D(s) = D_{\text{avg}} = \text{const}$$
$$K(s) = 1/\beta_{\text{avg}}^2$$
$$\beta_{\text{avg}} = R/Q$$
$$\rho = R$$

$$D_{\text{avg}} \approx \frac{R}{Q^2}$$

$$\alpha_c \approx \frac{< D >}{R} = \frac{1}{Q^2}$$

Approximate Dispersion Function

$$x'' + K(s)x = \frac{1}{\rho(s)} \frac{\Delta p}{p}$$ (the known differential equation)

- expanding right side in Fourier series
- solving DE for each series term
- keeping only first term to approximate x(s)
- deducing D(s) from x(s) = D(s) Δp/p
- **see Appendix** and Courant, Snyder reference

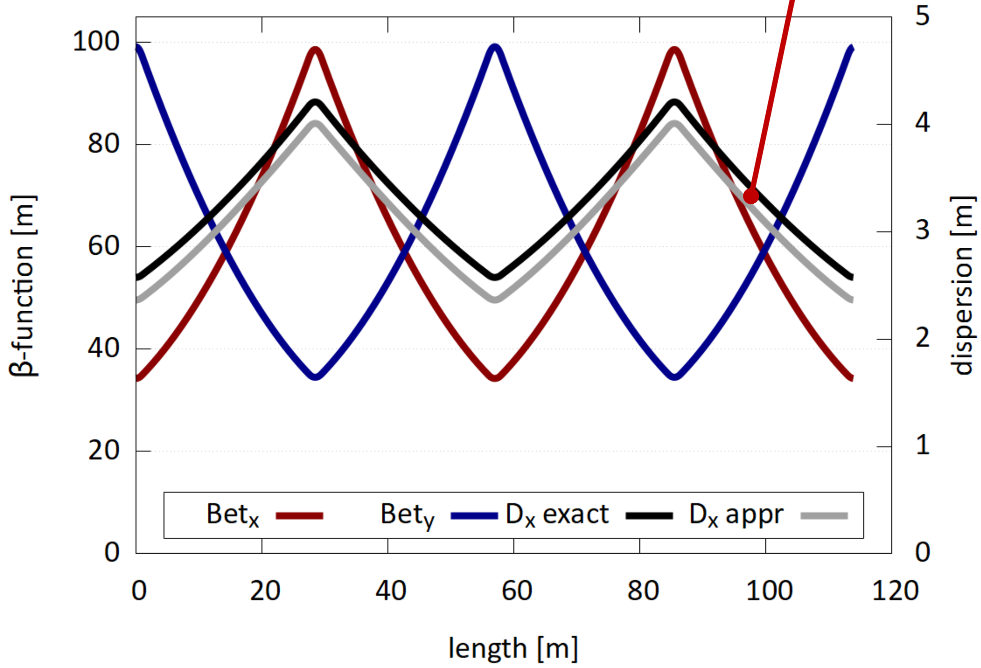
$$D_x(s) \approx \sqrt{\frac{R}{Q_x^3}} \cdot \sqrt{\beta_x(s)}$$

$\rightarrow$ Approximation can be used to estimate emittance in electron rings.

Approximate Dispersion Function :: Example

$$D_x(s) \approx \sqrt{\frac{R}{Q_x^3}} \cdot \sqrt{\beta_x(s)}$$

exact (black) and approximate dispersion function



example Ring:

- protons 500GeV, C=6270m
- 60deg / cell lattice, 110 cells
- ρ=840m, R=998m

Deriving approximate Momentum Compaction

change of path length through dispersion trajectory $x(s)$:

$$\Delta C = \int_0^C \frac{x}{\rho} ds = Q_x \int_0^{2\pi} \frac{\beta_x \cdot x}{\rho} d\theta$$

using series expansion $x(\theta)$, $\beta^{3/2}/\rho$, and retaining only n=0 term we obtain:

$$\Delta C = 2\pi Q_x^3 \, \frac{\Delta p}{p} \sum_{n=0}^{\infty} \frac{a_n^2}{Q_x^2 - n^2} \;\approx\; 2\pi Q_x a_0^2 \, \frac{\Delta p}{p}$$

with the previous result $a_0 \approx \sqrt{\frac{R}{Q_x^3}}$

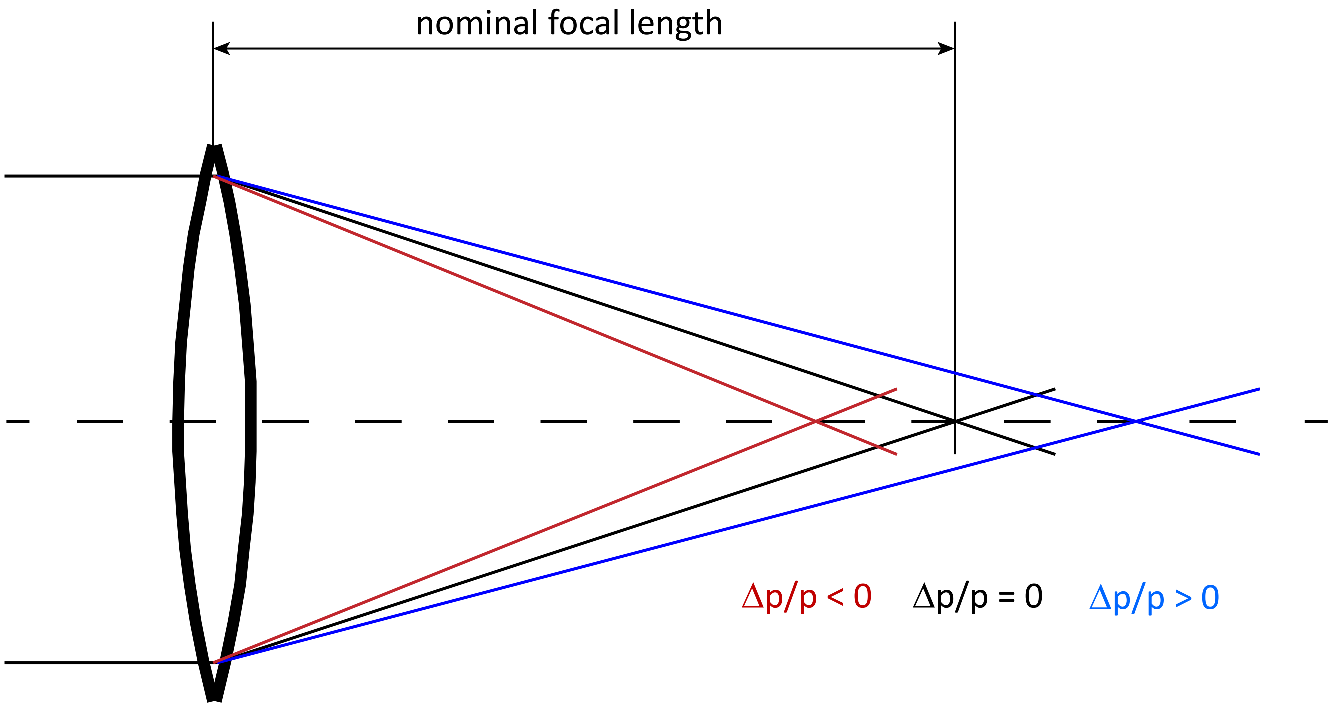
$$\alpha_c = \frac{\Delta C / C}{\Delta p / p} \approx \frac{1}{Q_x^2}$$

Next: Chromatic Focusing Error

- Focusing Error - What happens?
- Chromaticity
- Correction using Sextupole Magnets

Chromatic Errors

a spread of momentum leads to chromatic aberrations, similarly to aberrations of optical lenses:



Chromaticity

particles with momentum deviation are focused differently, leading to a shift of the betatron frequency

$$K = \frac{eg}{p} \quad dK = -\frac{eg\,dp}{p^2} = -K_0\frac{dp}{p}$$

Chromaticity ξ = change of tune per relative change of momentum:

$$\Delta Q = \xi\,\frac{\Delta p}{p_0}$$

integration over gradients around ring, beta-function as “sensitivity factor”:

$$\xi_x = -\frac{1}{4\pi}\oint K(s)\beta_x(s)ds$$

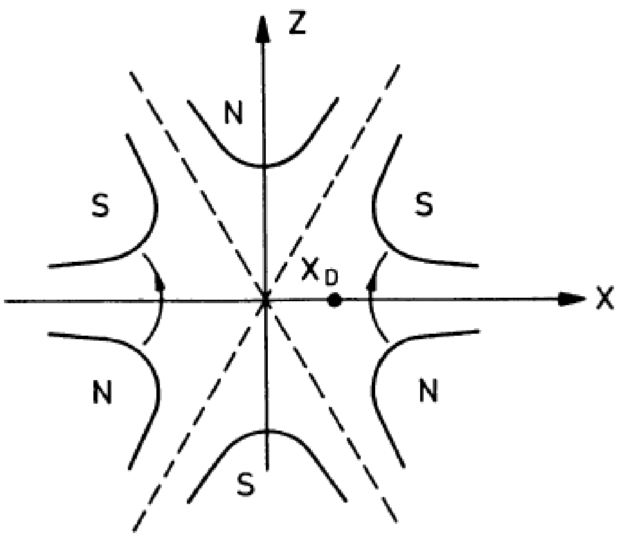
$\rightarrow$ “natural chromaticities” are always negative.

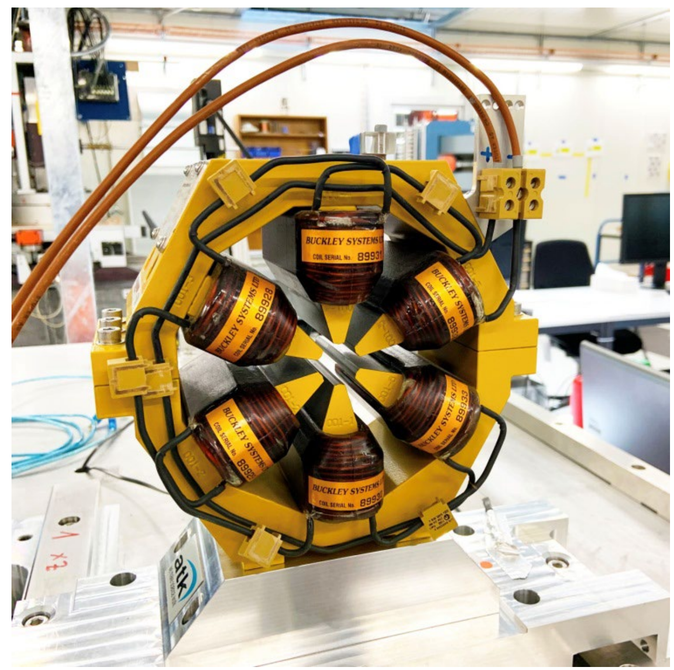
see Wiedemann sec. 15.4.1

Sextupol Magnet

Sextupoles are placed in a region of finite dispersion:
sort particles according to their energy deviation

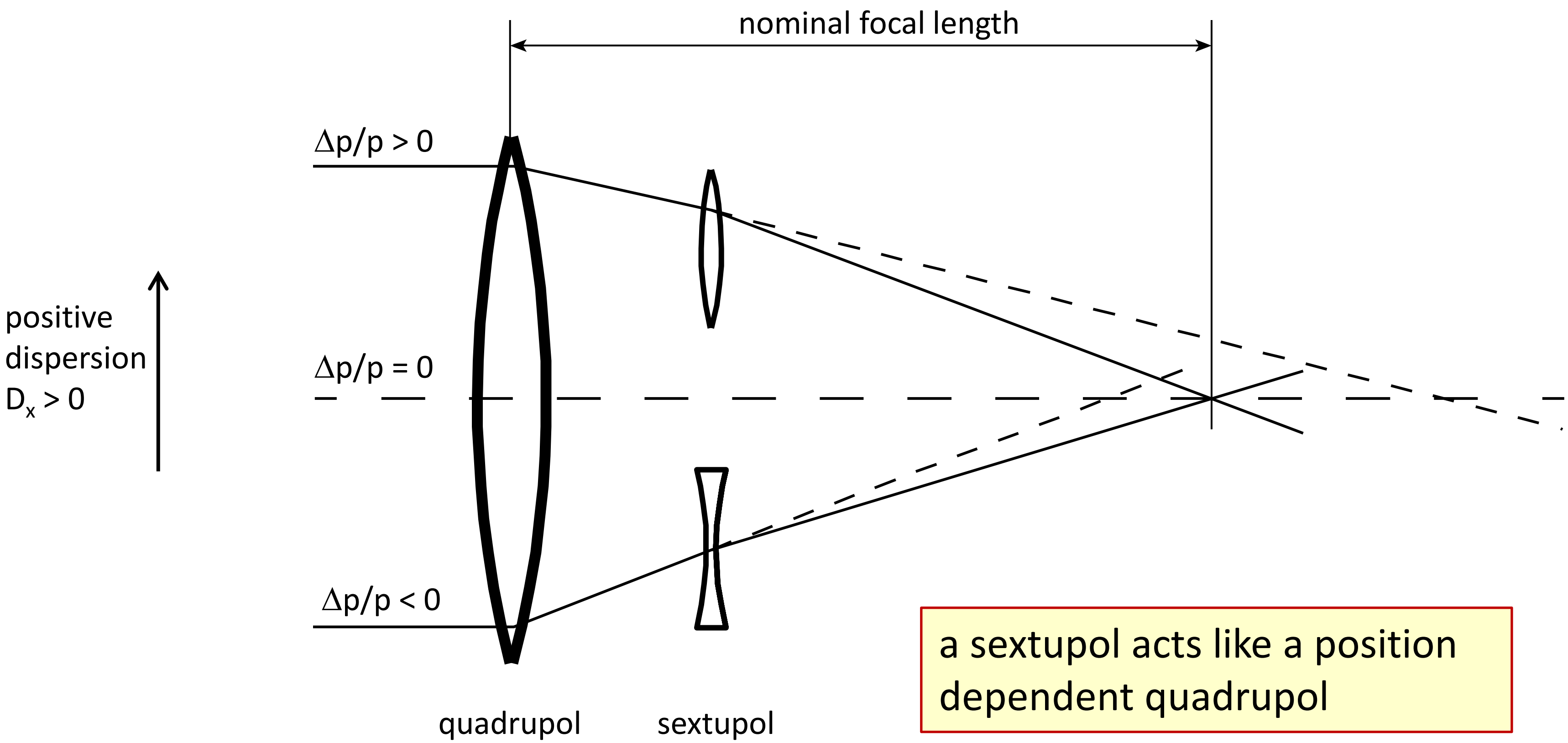
$$x_d = D(s) \cdot \frac{\Delta p}{p}$$





[PSI / SLS Sextupol]

Chromaticity – Correction using Sextupoles



a sextupol acts like a position dependent quadrupol

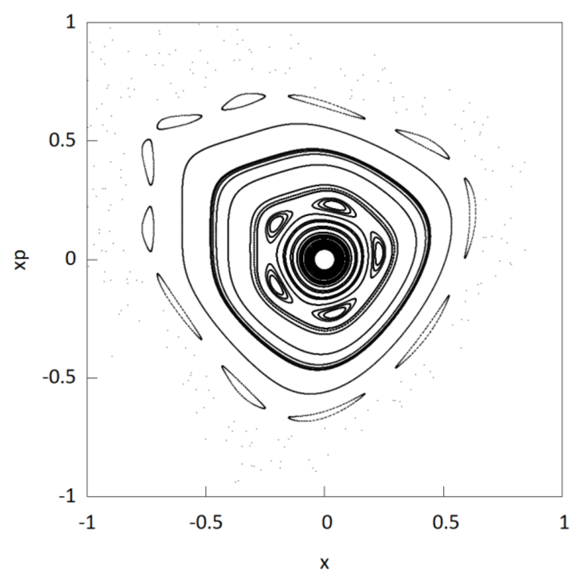
total chromaticity in a ring:

$$\xi_{\mathrm{tot}} = \frac{1}{4\pi} \oint (m(s)D(s) - K(s))\beta_x(s)ds$$

see Wiedemann sec. 15.4.2

Caution with Sextupoles

- while sextupoles can correct chromatic focusing errors, they are **nonlinear elements**
- nonlinear elements **drive resonances and reduce the dynamic aperture** of a ring, which must be carefully optimized when designing a ring



phase space portrait with sextupole kick

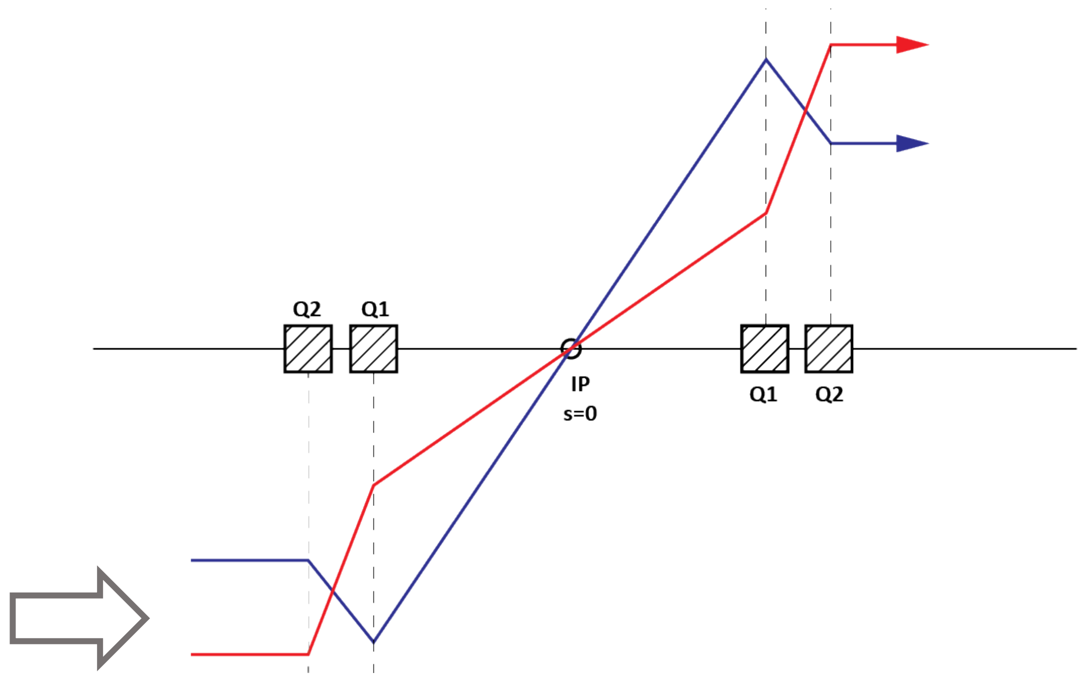
Next: Low Beta Insertion

- low Beta insertion
- another insertion in Appendix: dispersion suppressor

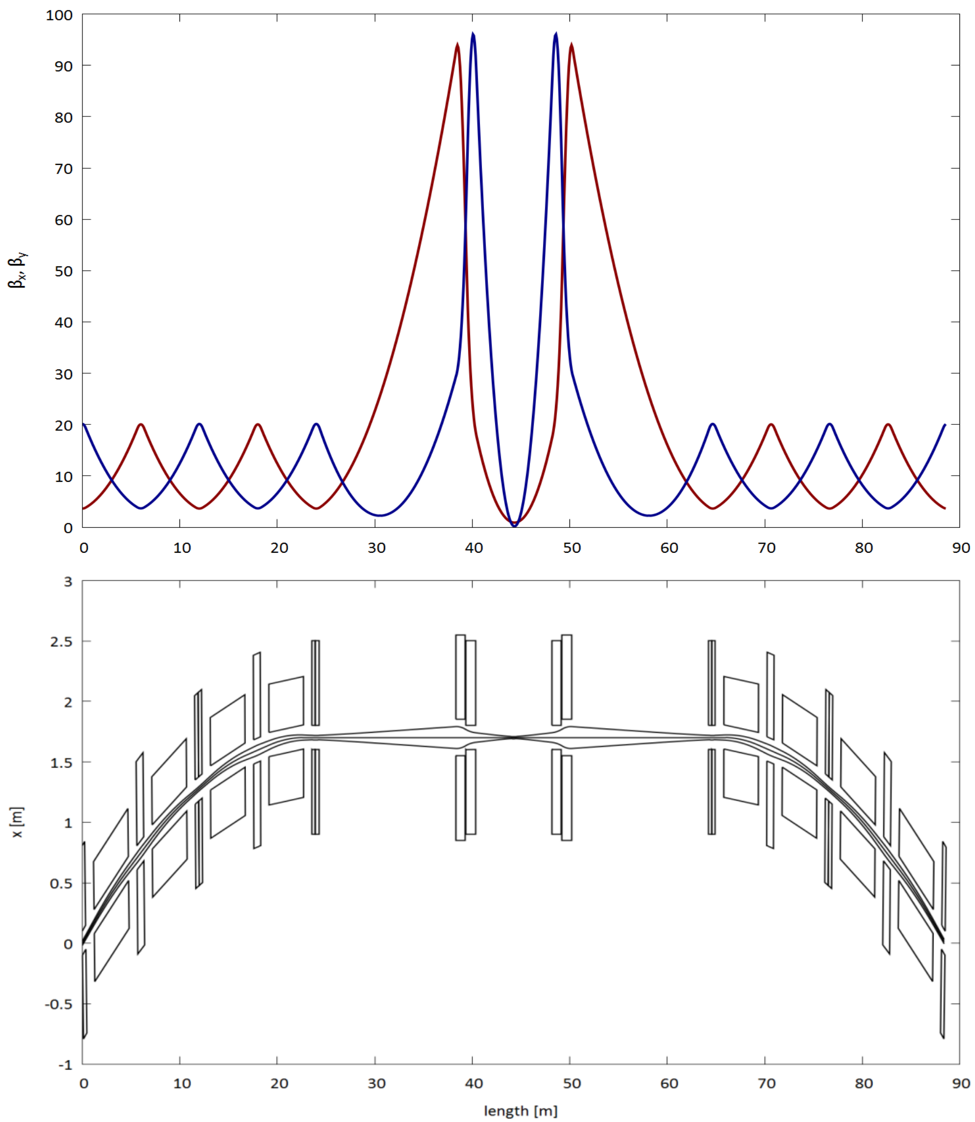
Low Beta Insertion

concept sketch: using a quadrupole doublet it is possible to focus particles in the horizontal and vertical planes simultaneously through the interaction point

incoming trajectories, parallel to reference orbit, in x,y



Low Beta Insertion



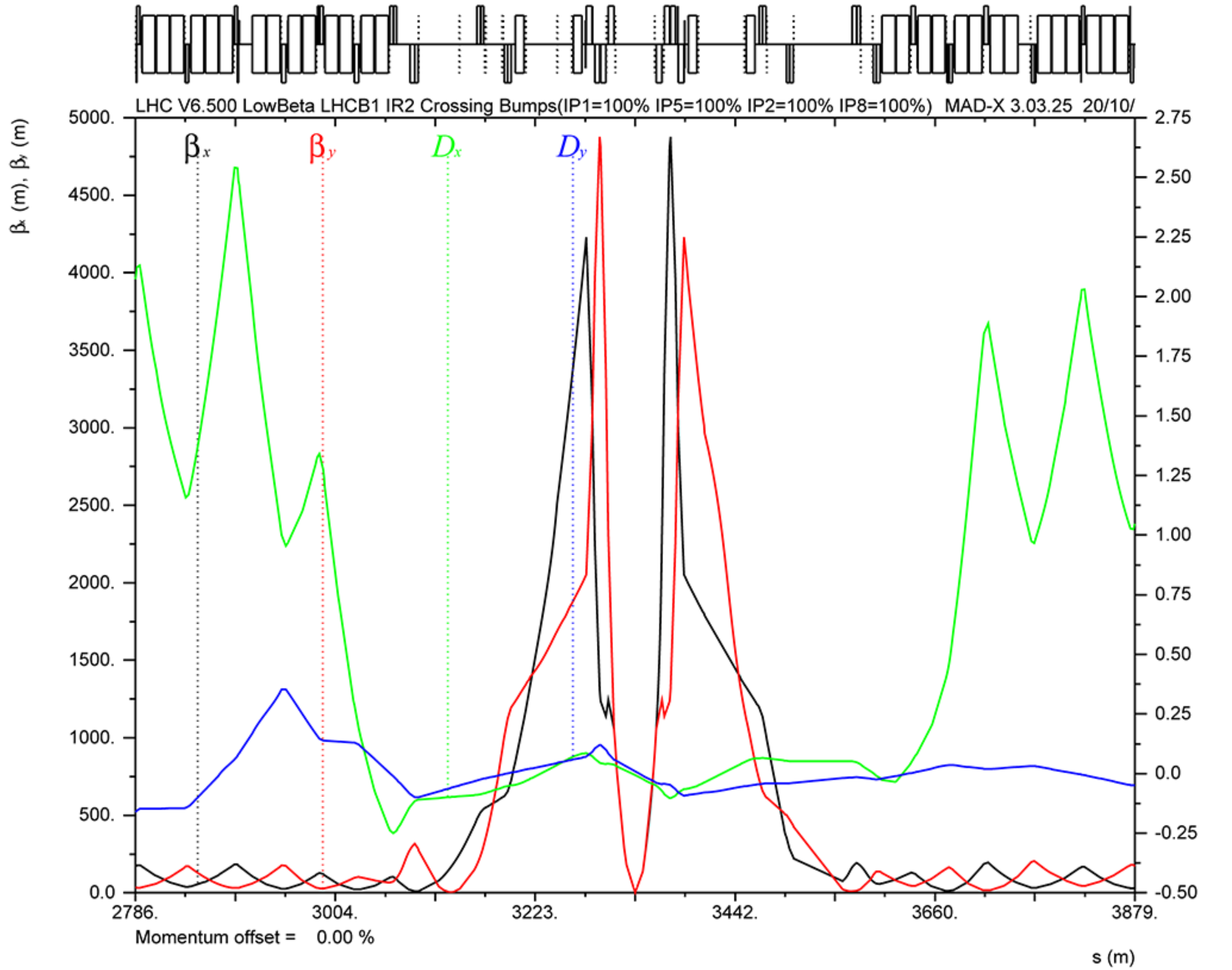
the most simple IR configuration

- doublet focusing
- large beta function in doublet
  $\rightarrow$ aperture limitation for ring

see also Wiedemann
sec. 10.2.4

Low Beta Insertion – Example of LHC

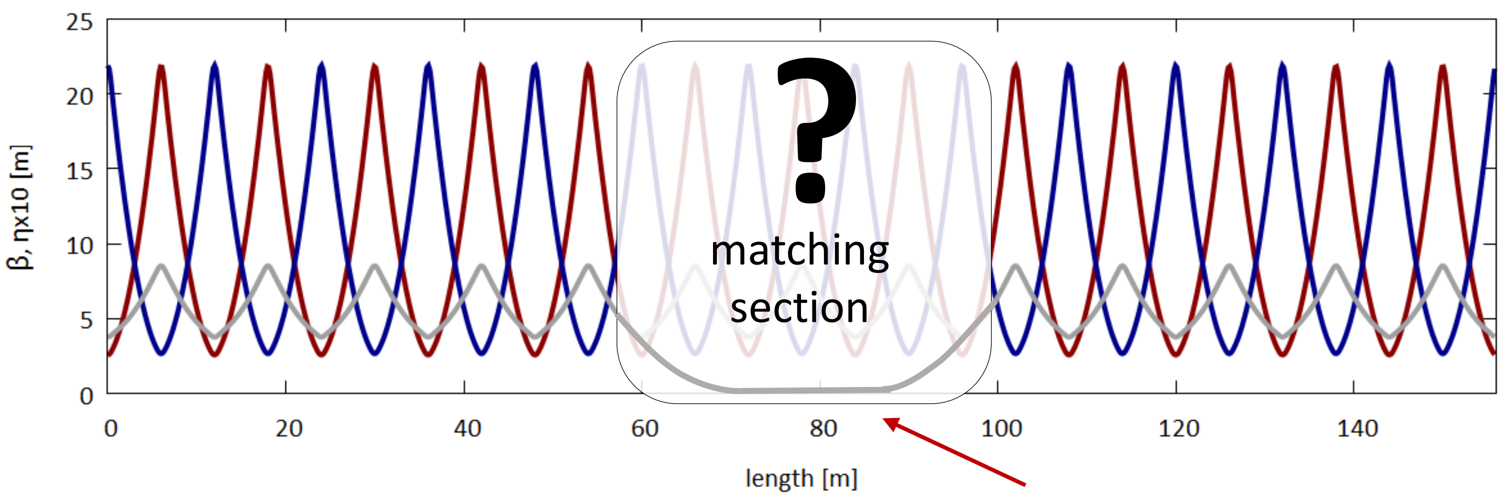
LHC interaction region with Low-Beta + D.S.



another insertion: Dispersion Suppressor

on average D is always positive in a ring, however it can be suppressed by special insertions

in certain situations dispersion must be suppressed / be small, e.g. interaction region, undulators in SR light sources



$$D = 0, \quad D' = 0$$

Dispersion Suppressor (continued)

one example approach to solve the problem:

in regular arc FODO cells with periodic dispersion:

$$\begin{pmatrix} D_c \\ D'_c \\ 1 \end{pmatrix} = \boldsymbol{M}_{\mathrm{c}} \begin{pmatrix} D_c \\ D'_c \\ 1 \end{pmatrix}$$

D matching section of *n* identical FODO cells:

$$\begin{pmatrix} D_c \\ D'_c \\ 1 \end{pmatrix} = \boldsymbol{M}_{\mathrm{s}}^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

solution without proof:

$$2\theta_{\mathrm{s}} \sin^2(n\mu_{\mathrm{c}}/2) = \theta_{\mathrm{c}}$$

$$\sin(n\mu_{\mathrm{c}}) = 0$$

θ_c and θ_s are bending angles in normal arc and matching section

possible solution:

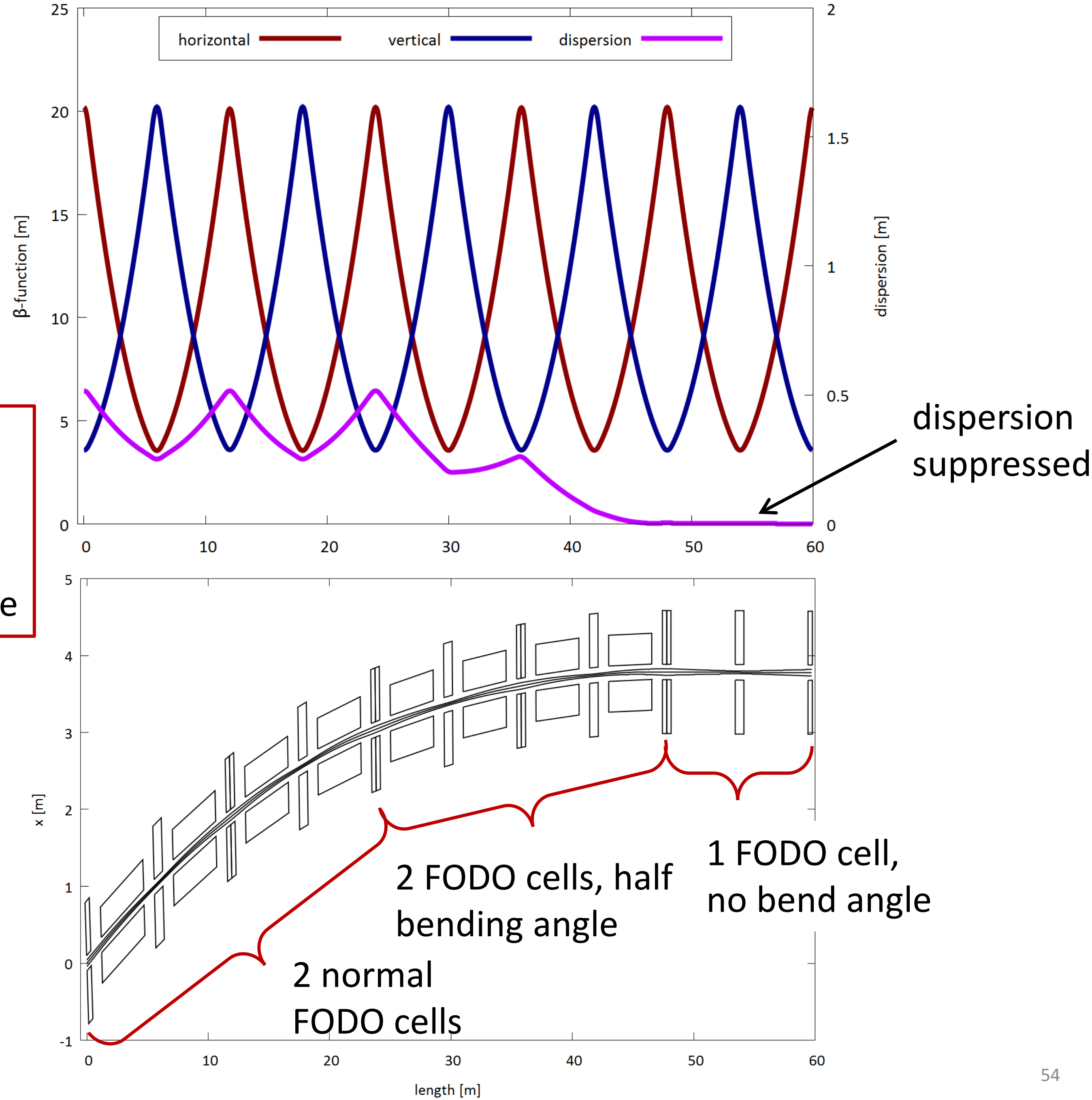
$n = 2$ ← 2 cells needed

$\mu_{\mathrm{c}} = 90\mathrm{deg}$ ← phase advance 90 degree p. cell

$\theta_{\mathrm{s}} = \frac{1}{2}\theta_{\mathrm{c}}$ ← deflection angle half of normal

Dispersion Supressor Example

- works only for φ_{cell}=90deg
- disadvantage: varying bend angle



see also Wiedemann sec. 10.2.4

Next: Summary Linear Beam Dynamics

What was discussed in Linear Dynamics II?

- Liouville theorem, phase space, emittance, beam distribution
- FODO cells, stability conditions, FODO with bending
- chromatic correction using sextupoles
- lattice insertions: dispersion suppressor, low beta insertion

Appendix: Approximate Dispersion Function

$$x'' + K(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p}$$ (the known differential equation)

constants of motion

$$x(s) = \underbrace{\sqrt{2J_x\beta_x}\cos(\varphi_x - \varphi_0)} + \underbrace{D_x\frac{\Delta p}{p}}$$

homogeneous and particular solution

use: $$x(s) = z(\theta)\beta_x^{\frac{1}{2}},\ \theta = \frac{1}{Q_x}\int_0^s \frac{ds'}{\beta_x(s')}$$

new DE: $$\underbrace{z'' + Q_x^2 z} = \underbrace{Q_x^2\frac{\beta_x^{\frac{3}{2}}}{\rho}\frac{\Delta p}{p}}$$

harmonic oscillator and driving term

Appendix: Solution by Fourier Expansion

$$z'' + Q_x^2 z = Q_x^2 \frac{\beta_x^{\frac{3}{2}}}{\rho} \frac{\Delta p}{p}$$ ← driving term

expand driving term:

$$\frac{\beta_x^{\frac{3}{2}}}{\rho} = \sum_{n=0}^{\infty} a_n e^{in\theta}, \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta_x^{\frac{3}{2}}}{\rho} e^{-in\theta} d\theta$$

insertion exp($in\theta$) in DE yields trajectory for $\Delta p/p$:

$$x(\theta) = \beta_x^{\frac{1}{2}} Q_x^2 \frac{\Delta p}{p} \sum_{n=0}^{\infty} \frac{a_n}{Q_x^2 - n^2} e^{in\theta}$$

large terms for

- n=0, as $\beta^{3/2}/\rho$ always positive, thus a_0 large
- and $Q_x \approx n$ (resonant term)

Fourier Expansion of Dispersion (continued)

use only a_0 as dominating term:

$$x = D_x \frac{\Delta p}{p} \approx a_0 \beta_x^{\frac{1}{2}} \frac{\Delta p}{p} \longrightarrow D_x \approx a_0 \beta_x^{\frac{1}{2}}$$

and:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta^{\frac{3}{2}}}{\rho} e^{-in\theta} d\theta = \frac{1}{2\pi Q_x} \int_0^{C} \frac{\beta^{\frac{1}{2}}}{\rho} e^{-in\theta} ds$$

$$= \frac{1}{Q_x} \left\langle \beta_x^{\frac{1}{2}} \right\rangle_{\text{magnets}}, \quad \text{use :} \quad \beta_x \approx \frac{R}{Q_x} = \text{const}$$

thus:

$$D_x(s) \approx \sqrt{\frac{R}{Q_x^3}} \cdot \sqrt{\beta_x(s)}$$