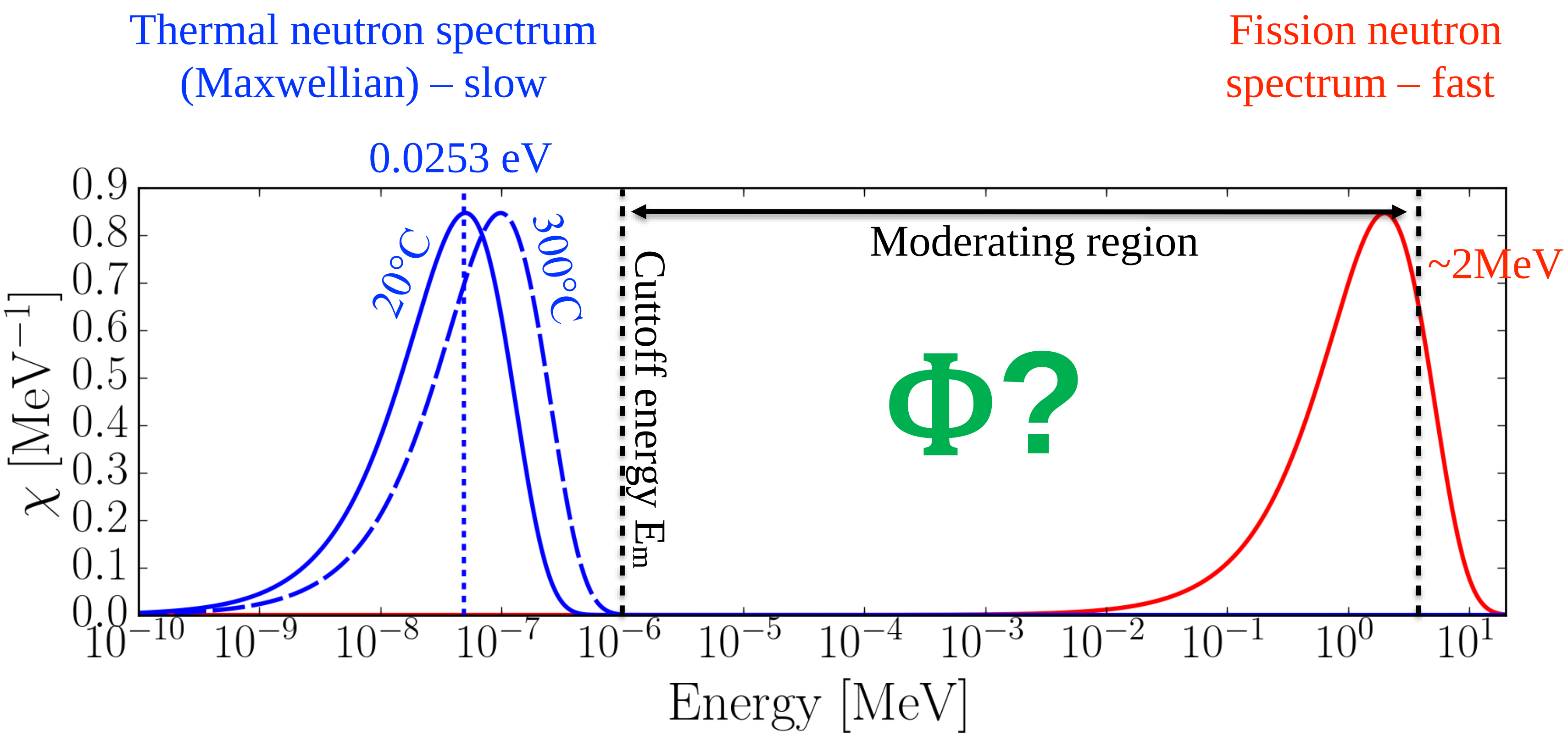
EPFL

Broad topic	Lecture title
Basic principles of NPP	Introduction / Review of nuclear physics
	Interaction of neutrons with matter
	Nuclear fission
	Fundamentals of nuclear reactors
	LWR plants
Modeling the beast	The diffusion of neutrons - Part 1
	The diffusion of neutrons - Part 2
	Neutron moderation without absorption
	Neutron moderation with absorption
	Multigroup theory
	Element of lattice physics
	Neutron kinetics
	Depletion
Reactor Concepts Zoo	Advanced LWR technology
	Breeding and LFR
	AGR, HTGR
	Channels, MSR and thorium fuel
Review session	

- Energy distribution of neutrons in a thermal reactor
- Study of an elastic collision
- Slowing down equations
- Fermi age theory

- Till now, we have discussed the spatial behaviour of monoenergetic neutrons with **appropriately averaged** cross-sections…

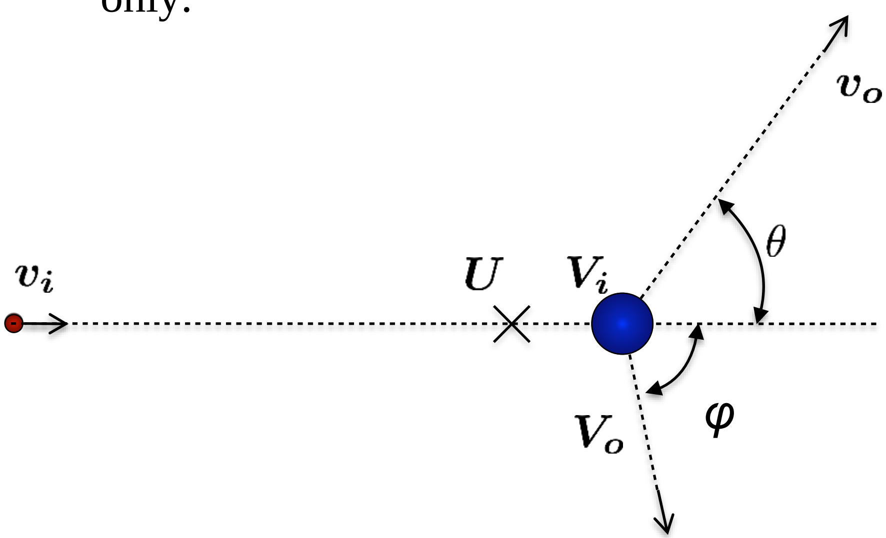
- Fission neutrons have E~2 MeV. Successive collisions with nuclei result in neutron energy losses and reduction of the neutron energy by several orders of magnitude



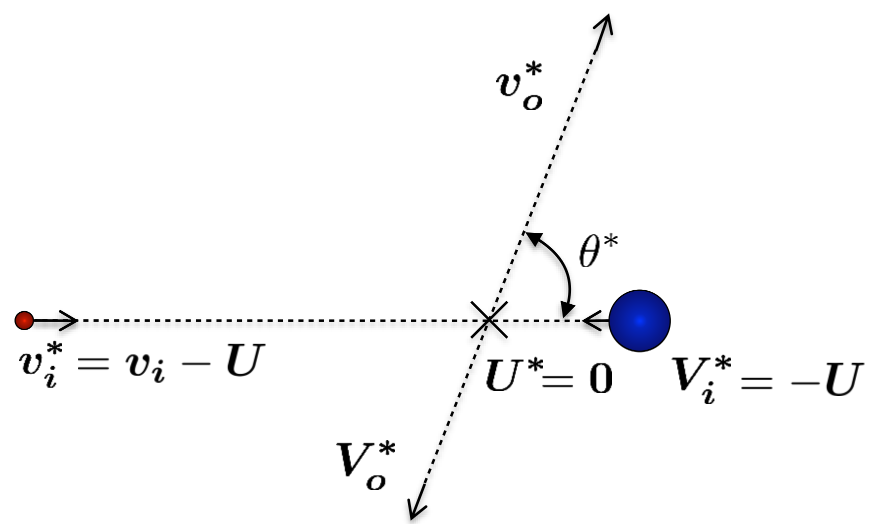
Elastic collision

- Most important slowing-down mechanism: elastic scattering by moderator nuclei
- Inelastic scattering also plays a role, but only for fast neutrons (E ≥ 1 MeV)
- Laboratory system: **nucleus A is at rest**. 2 angles needed to describe the collision
- Center of mass system: **center of mass is at rest**. 1 angle only.

		in, out
neutron –	lab :	$v_i,\ v_o$
	CoM :	$v_i^*,\ v_o^*$
Nucleus –	lab :	$V_i,\ V_o$
	CoM :	$V_i^*,\ V_o^*$
× –	lab :	U
	CoM :	U^*



Laboratory system

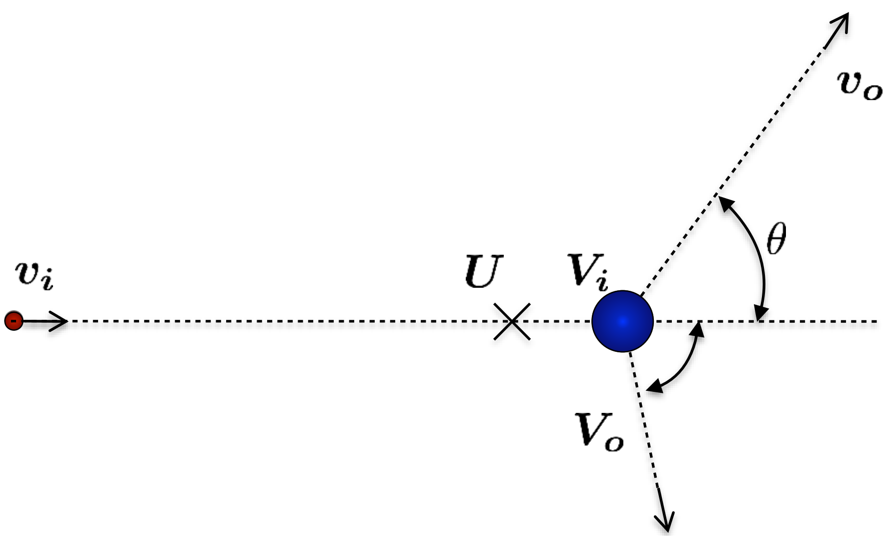


Center-of-mass system

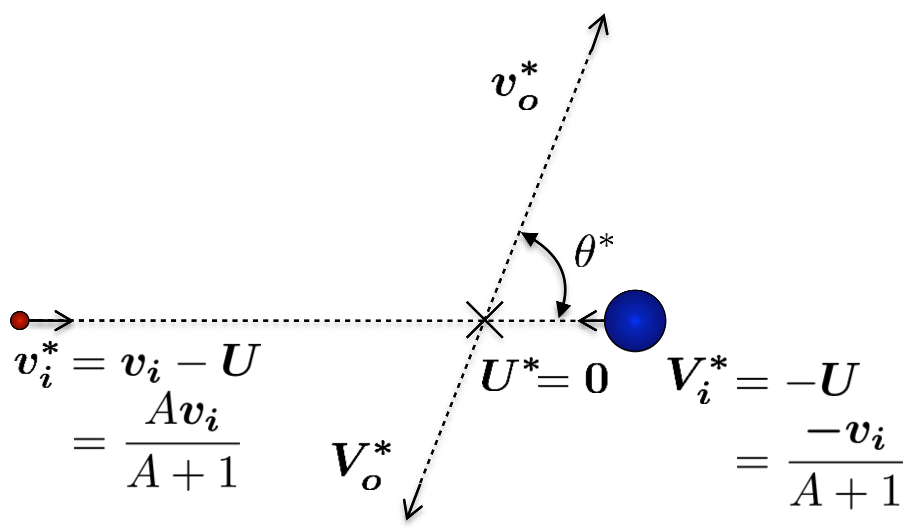
- CoM is at rest: $m\,\boldsymbol{v}_i^* + M\,\boldsymbol{V}_i^* = 0 \;\Rightarrow\; \boldsymbol{v}_i^* + A\,\boldsymbol{V}_i^* = 0$
  $$\Rightarrow (\boldsymbol{v_i} - \boldsymbol{U}) - A\boldsymbol{U} = 0$$
  $$\Rightarrow \boldsymbol{U} = \frac{\boldsymbol{v_i}}{A+1}$$
- For neutron: $\boldsymbol{v}_i^* = \boldsymbol{v_i} - \boldsymbol{U} = \dfrac{A\boldsymbol{v_i}}{A+1}$
- For nucleus: $\boldsymbol{V}_i^* = -\boldsymbol{U} = -\dfrac{\boldsymbol{v_i}}{A+1}$

		in, out
neutron –	lab :	$\boldsymbol{v_i},\ \boldsymbol{v_o}$
	CoM :	$\boldsymbol{v}_i^*,\ \boldsymbol{v}_o^*$
Nucleus –	lab :	$\boldsymbol{V_i},\ \boldsymbol{V_o}$
	CoM :	$\boldsymbol{V}_i^*,\ \boldsymbol{V}_o^*$
× –	lab :	$\boldsymbol{U}$
	CoM :	$\boldsymbol{U}^*$

$A = {}^{M}/_{m}$



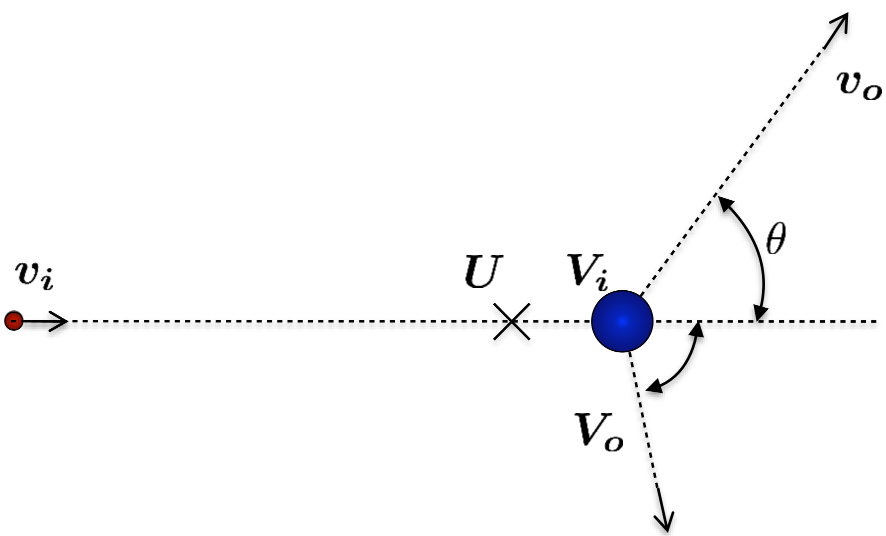
Laboratory system



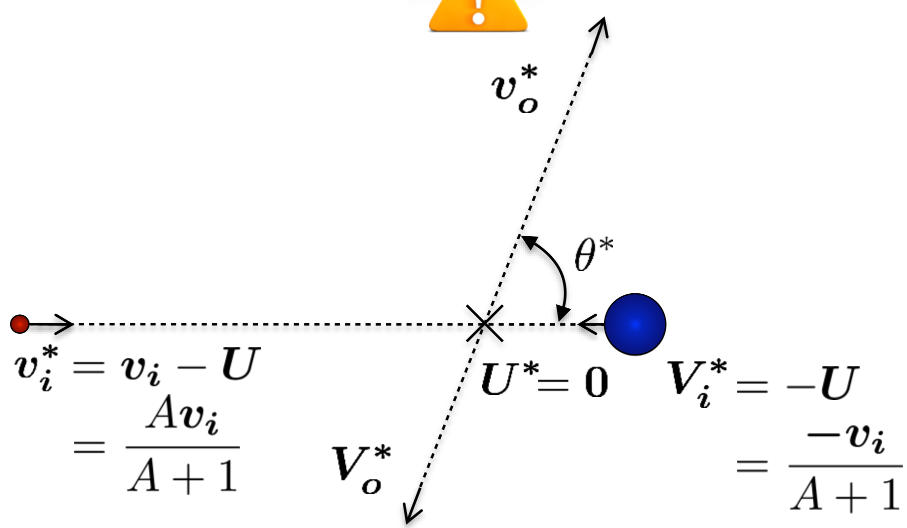
Center-of-mass system

$$\boldsymbol{v_i^*} = \boldsymbol{v_i} - \boldsymbol{U} = \frac{A\boldsymbol{v_i}}{A+1} \qquad \boldsymbol{V_i^*} = -\boldsymbol{U} = -\frac{\boldsymbol{v_i}}{A+1}$$

- In CoM: $v_i^{*2} + AV_i^{*2} = v_o^{*2} + AV_o^{*2}$ Conservation of energy
- From conservation of momentum: $v_i^* = -AV_i^*$ and $v_o^* = -AV_o^*$
- Finally $v_i^* = v_o^*$ and $V_i^* = V_o^*$
- The velocities remain the same in the CoM (only the direction changes)



Laboratory system



Center-of-mass system

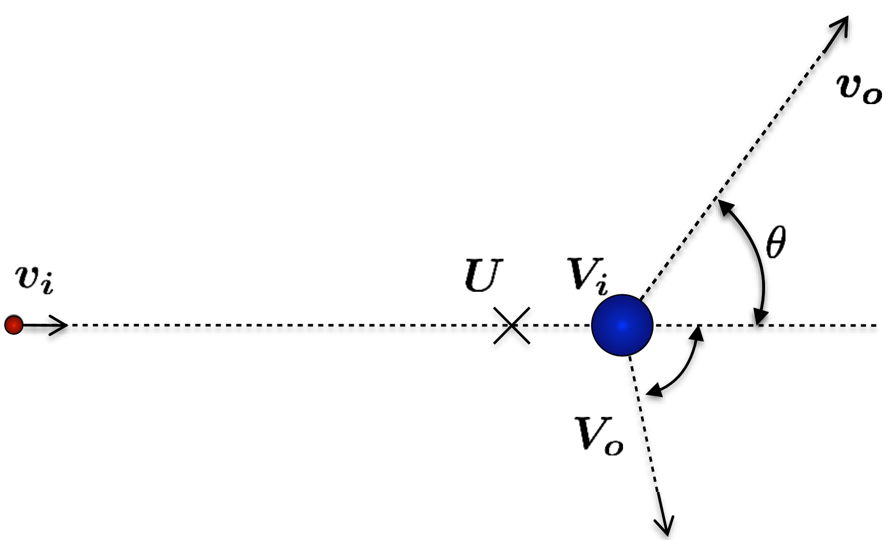
Fractional energy loss in an elastic collision

- Law of cosines: $v_o{}^2 = v_i^{*2} + U^2 + 2v_i^* U cos(\theta^*)$

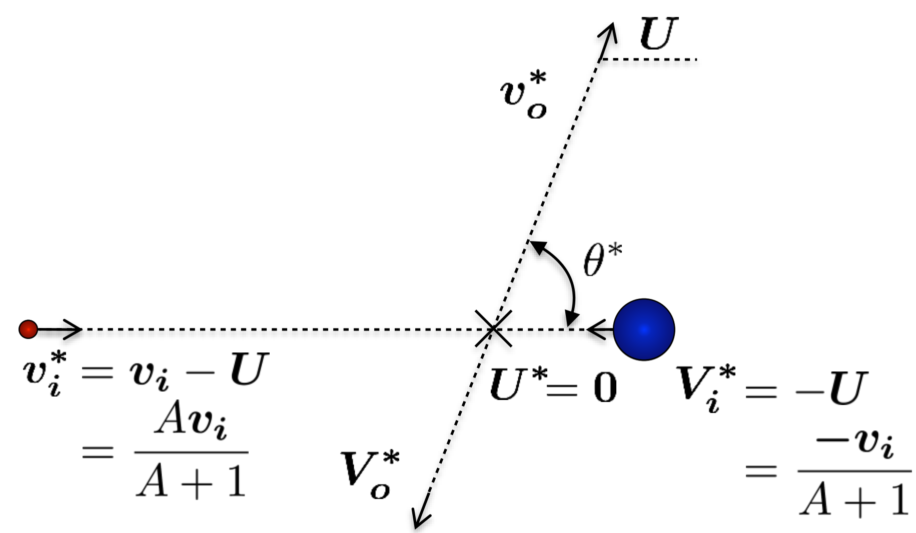
- Fractional energy loss in a single elastic collision in laboratory:

$$\frac{E_i - E_o}{E_i} = 1 - \frac{v_o{}^2}{v_i{}^2} = 1 - \frac{A^2 + 2A\cos\theta^* + 1}{(A+1)^2} = (1-\cos\theta^*)\frac{1-\alpha}{2}$$

where $\alpha = \left(\frac{A-1}{A+1}\right)^2$



Laboratory system



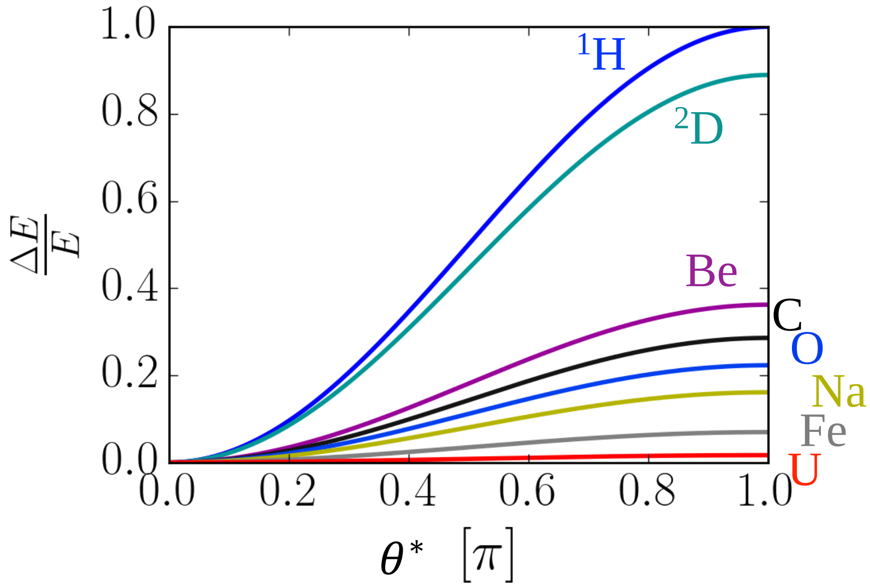
Center-of-mass system

- Law of cosines: $v_o{}^2 = v_i^{*2} + U^2 + 2v_i^* U cos(\theta^*)$

- Fractional energy loss in a single elastic collision in laboratory:

$$\frac{\Delta E}{E_i} = \frac{E_i - E_o}{E_i} = 1 - \frac{v_o{}^2}{v_i{}^2} = 1 - \frac{A^2 + 2A\cos\theta^* + 1}{(A+1)^2} = (1 - \cos\theta^*)\,\frac{1-\alpha}{2}$$

where $\alpha = \left(\frac{A-1}{A+1}\right)^2$



Or (useful later) :

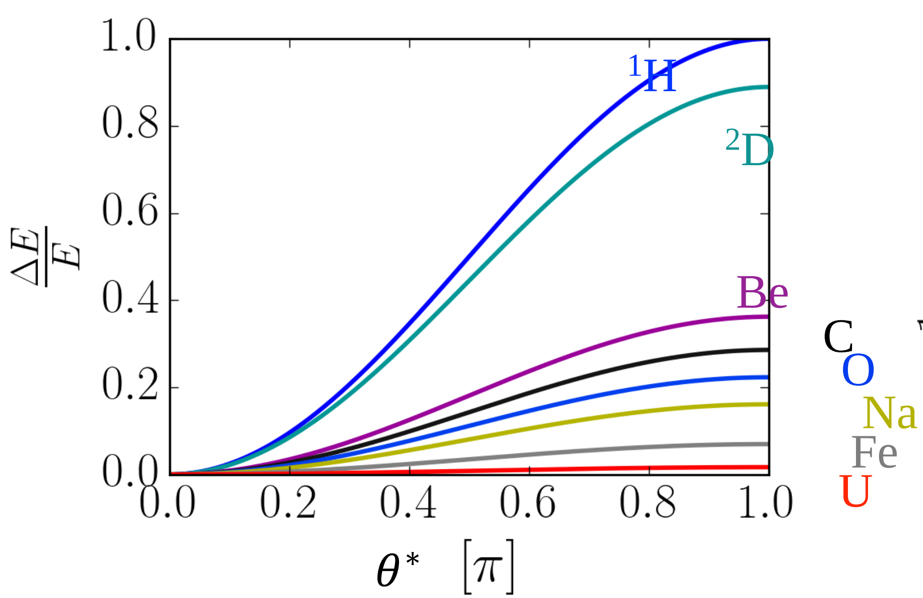
$$\frac{E_o}{E_i} = \frac{v_o{}^2}{v_i{}^2} = \frac{A^2 + 2A\cos\theta^* + 1}{(A+1)^2}$$

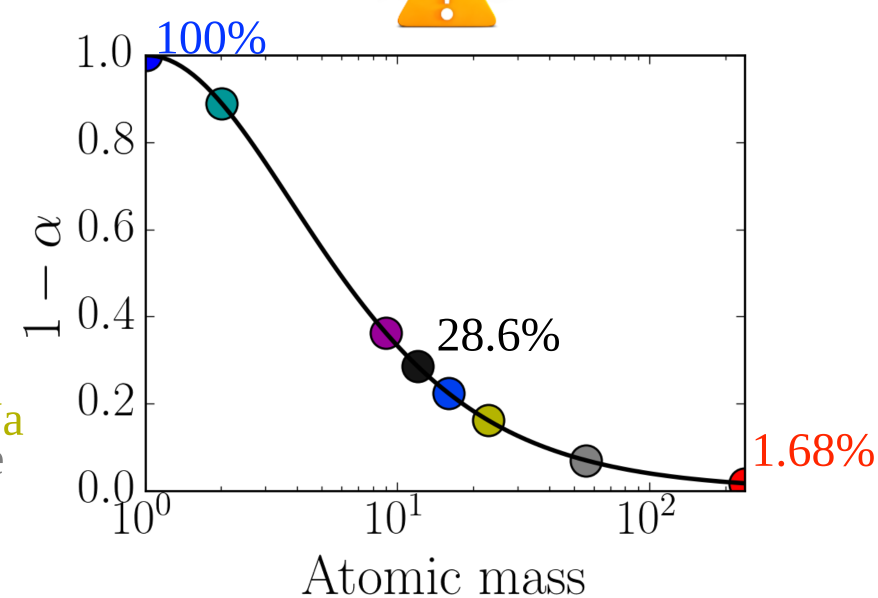
Fractional Energy Loss in an Elastic Collision

- Fractional energy loss:

$$\frac{\Delta E}{E_i} = (1 - \cos\theta^*)\frac{1-\alpha}{2} \qquad \text{where} \qquad \alpha = \left(\frac{A-1}{A+1}\right)^2$$

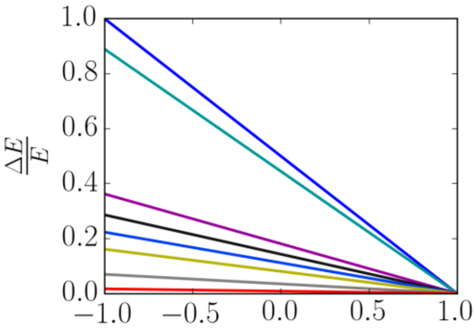
- Energy loss in a single elastic collision is proportional to scattering angle
- When θ* = 0 energy loss is minimum: $\Delta E = 0$
- When θ* = π energy loss is maximum: $\Delta E/E_i = 1 - \alpha \;\Rightarrow\; E_o = \alpha E_i$
- Maximum fractional energy loss in a single elastic collision is $1 - \alpha$





‘light’ notation: $(E_i, E_o) \iff (E, E')$

- Probability distribution function $P_1(E \to E')$:
  $P_1(E \to E')\mathrm{d}E$ – probability that a neutron with the laboratory energy E will have after collision an energy between E’ and E’+ dE'.

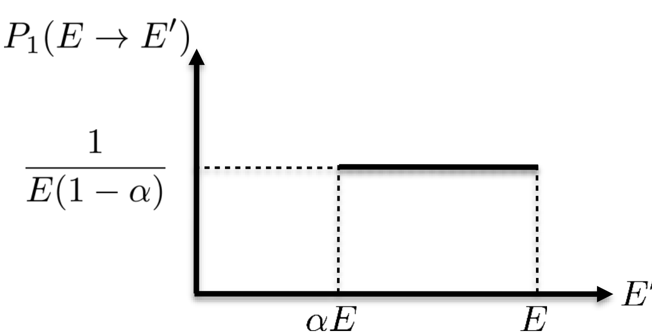


- In the majority of cases, scattering is isotropic in CoM system
- Using as variable $\mu = \cos\theta^*$, no. of n’s scattered between $[\mu, \mu + \mathrm{d}\mu]$ is $\propto \mathrm{d}\mu$
- $\theta^* \in [0, \pi] \Rightarrow \mu \in [-1,1]$ then the max. width: $\Delta\mu = 2 \Rightarrow$ fraction betn. $[\mu, \mu + \mathrm{d}\mu]$ is $\mathrm{d}\mu/2$
- Differentiating $\frac{E'}{E} = \frac{A^2 + 2A\mu + 1}{(A+1)^2} \Rightarrow \frac{\mathrm{d}E'}{E} = \frac{2A}{(A+1)^2}\mathrm{d}\mu$

$$\Rightarrow \frac{\mathrm{d}\mu}{2} = \frac{(A+1)^2}{4A}\frac{\mathrm{d}E'}{E} = \frac{\mathrm{d}E'}{E(1-\alpha)}, \quad \text{where } \alpha = \left(\frac{A-1}{A+1}\right)^2$$

- Thus, probability for a neutron to have an energy in [E’, E’+dE’]:

$$P_1(E \to E') = \begin{cases} \frac{1}{E(1-\alpha)}, & \alpha E < E' < E \\ 0, & 0 < E' < \alpha E \end{cases}$$



- Thus, probability for a neutron to have an energy in [E', E'+dE'] :

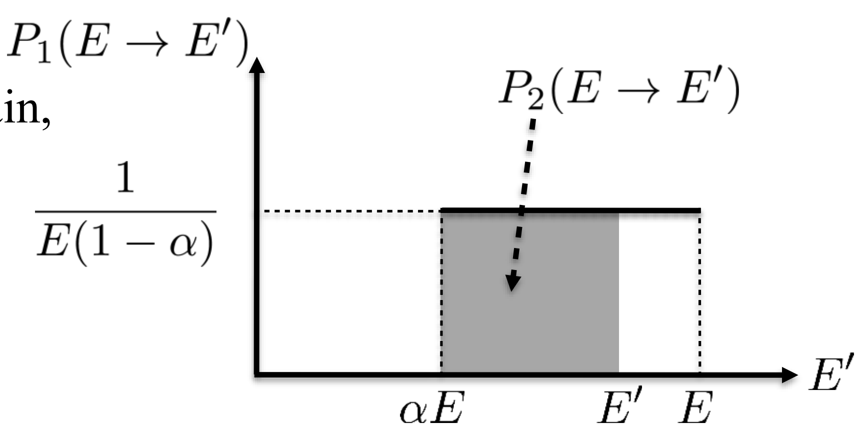
$$P_1(E \to E') = \begin{cases} \frac{1}{E(1-\alpha)}, & \alpha E < E' < E \\ 0, & 0 < E' < \alpha E \end{cases}$$

- Probability distribution function $P_2(E \to E')$:
$P_2(E \to E')\mathrm{d}E'$ – probability that a neutron with the laboratory energy E will have after collision, an energy ***below*** E'.

$$P_2(E \to E') = \int_{\alpha E}^{E'} P_1(E \to E'')\mathrm{d}E'' = \begin{cases} \frac{E' - \alpha E}{E(1-\alpha)}, & \alpha E < E' < E \\ 0, & \text{otherwise} \end{cases}$$

- $P_2(E \to E') = 1 \text{ for } E = E'$

  That E' lies between E and αE is certain, P2 decreases linearly



- For the energy band $[E, E+\mathrm{d}E]$, the steady state neutron balance equation is:

$$\nabla \cdot (\vec{J}\,\mathrm{d}E) + \Sigma_a(\phi\,\mathrm{d}E) = Q\,\mathrm{d}E$$

- $Q\,\mathrm{d}E$ is the total sources between $[E, E+\mathrm{d}E]$

→ "True" (fission, isotopic sources,…), as well as those resulting from slowing down (neutrons of energy $> E$ are scattered into the band $[E, E+\mathrm{d}E]$)

- Considering the n's in $[E', E'+\mathrm{d}E']$, scattering rate is

$$\mathrm{d}R_s = \Sigma_s(E')[\phi(\vec{r}, E')\,\mathrm{d}E']$$

- No. scattered with an energy below E is: $P_2(E' \to E)\,[\Sigma_s(E')\,\phi(\vec{r}, E')\,\mathrm{d}E']$

- Total no. scattered below E at $\vec{r}$: $$q(\vec{r}, E) = \int_E^{\infty} P_2(E' \to E)\,[\Sigma_s(E')\,\phi(\vec{r}, E')\,\mathrm{d}E']$$

→ ***Slowing-down*** source [$\mathrm{cm}^3\mathrm{s}^{-1}$]

- Assuming a single scattering nucleus
- With $P_2(E \to E') = \begin{cases} \dfrac{E' - \alpha E}{E(1-\alpha)}, & \alpha E < E' < E \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow q(\vec{r}, E) = \int_E^{E/\alpha} \frac{E - \alpha E'}{(1-\alpha)E'} \left[\Sigma_s(E')\,\phi(\vec{r}, E')\,\mathrm{d}E'\right] \quad (1)$$

- Difference $[q(\vec{r}, E + \mathrm{d}E) - q(\vec{r}, E)]$ gives slowing-down source in band $[E, E + \mathrm{d}E]$
- Thus, neutron balance equation:

$$\nabla \cdot \left(\vec{J}(\vec{r}, E)dE\right) + \Sigma_a(E)\,\phi(\vec{r}, E)\,dE = [q(\vec{r}, E + dE) - q(\vec{r}, E)] + Q_f(\vec{r}, E)dE$$

$$\nabla \cdot \left(\vec{J}(\vec{r}, E)\right) + \Sigma_a(E)\,\phi(\vec{r}, E) = \frac{\partial}{\partial E} q(\vec{r}, E) + Q_f(\vec{r}, E) \quad (2)$$

Fundamental Slowing-down Equations

- Considering Eq. (1)

$$q(\vec{r}, E) = \int_E^{E/\alpha} \frac{E - \alpha E'}{(1-\alpha)E'} \left[\Sigma_s(E')\,\phi(\vec{r}, E')\,\mathrm{d}E'\right]$$

$$\Rightarrow \quad \frac{\partial q}{\partial E} = \int_E^{E/\alpha} \frac{\Sigma_s(E')\,\phi(\vec{r}, E')}{(1-\alpha)E'}\,\mathrm{d}E' + \frac{1}{\alpha}\left[\frac{E - \alpha E'}{(1-\alpha)E'}\,\Sigma_s(E')\,\phi(\vec{r}, E')\right]_{E'=E/\alpha} - 1\left[\frac{E - \alpha E'}{(1-\alpha)E'}\,\Sigma_s(E')\,\phi(\vec{r}, E')\right]_{E'=E}$$

$$\Rightarrow \quad \frac{\partial q}{\partial E} = \int_E^{E/\alpha} \frac{\Sigma_s(E')\,\phi(\vec{r}, E')}{(1-\alpha)E'}\,\mathrm{d}E' - \Sigma_s(E)\,\phi(\vec{r}, E)$$

Math break

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(x,t)\,\mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\mathrm{d}f(x,t)}{\mathrm{d}t}\,\mathrm{d}x + f(b(t),t)\,\frac{\mathrm{d}b(t)}{\mathrm{d}t} - f(a(t),t)\,\frac{\mathrm{d}a(t)}{\mathrm{d}t}$$

- Considering Eq. (2)

$$\nabla \cdot \vec{J}(\vec{r},E) + \Sigma_a(E)\,\phi(\vec{r},E) = \int_E^{E/\alpha} \frac{\Sigma_s(E')\,\phi(\vec{r},E')}{(1-\alpha)E'}\,dE' - \Sigma_s(E)\,\phi(\vec{r},E) + Q_f(\vec{r},E)$$

- And introducing the Fick's law

$$\nabla \cdot [D(E)\nabla\phi(\vec{r},E)] - [\Sigma_a(E) + \Sigma_s(E)]\,\phi(\vec{r},E) + \int_E^{E/\alpha} \frac{\Sigma_s(E')\,\phi(\vec{r},E')}{(1-\alpha)E'}\,dE' + Q_f(\vec{r},E) = 0$$

- … Diffusion Equation for the band $[E, E + \mathrm{d}E]$ → yields the energy dependent scalar $\phi(\vec{r}, E)$ flux

EPFL

- In the more general case of a mixture of k nuclei, each with a nuclide concentration N_k

- The slowing down source is the combination of the slowing down source for each separate nucleus

$$q(E) = \sum_k \int_E^{E/\alpha_k} \frac{E - \alpha_k E'}{(1-\alpha_k)E'} N_k \sigma_s(E') \phi(E') dE' \qquad (1)$$

- And finally for the balance equation:

leakage term

$$\nabla \cdot \vec{J}(\vec{r}, E) + \Sigma_t(E)\, \phi(\vec{r}, E) = \sum_k \int_E^{E/\alpha_k} \frac{N_k \sigma_s(E') \phi(E')}{(1-\alpha_k)E'} dE' + Q_f(\vec{r}, E) \qquad (2)$$

total collision term *scattering source* *Fission source*

- NB: $\Sigma_t(E) = N_k \sigma_t(E')$

In developing the slowing-down theory, it is interesting to consider an academic case:

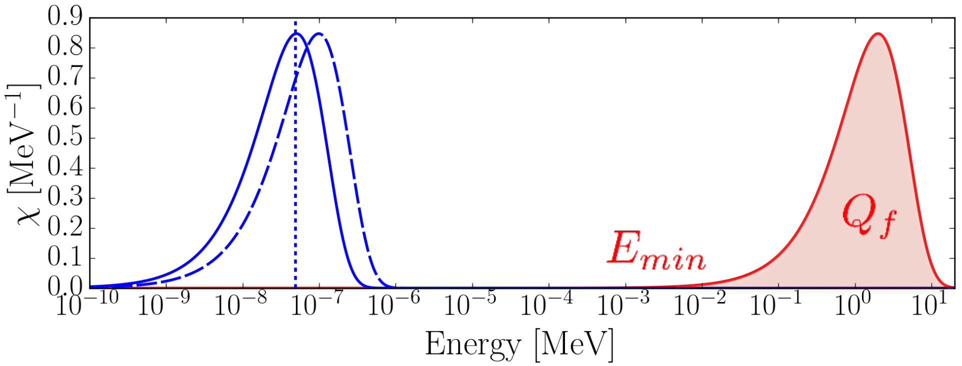
- ***infinite*** homogeneous hydrogenous medium (no dependence on position)
- ***no absorption***
- ***steady state*** problem

$$\nabla \cdot \cancel{\vec{J}(\vec{r}, E)} + \cancel{\Sigma_a(E)}\, \phi(\vec{r}, E) = \frac{\partial}{\partial E}\, q(\vec{r}, E) + Q(\vec{r}, E)$$

$$q(\vec{r}, E) = \int_E^{E/\alpha} \frac{E - \alpha E'}{(1-\alpha)E'}\, \Sigma_s(E')\, \phi(\vec{r}, E')\, \mathrm{d}E'$$

- Very simple neutron balance equation: $$\frac{\mathrm{d}q(E)}{\mathrm{d}E} + Q(E) = 0$$

  In general, Q is the fission-source density (fission spectrum)



- Integrating
  $$[q(\infty) - q(E)] + \int_E^\infty Q(E')\,\mathrm{d}E' = 0$$
  → $$\lim_{E\to\infty} q(E) = 0$$
  → for $E < E_{min}$, $\int_E^\infty Q(E')\,\mathrm{d}E' = Q_f$ *total fission source*

  $\Rightarrow$ for $E < E_{min}$, $q(E) = Q_f$ *constant slowing down source*

- In absence of absorption ($\Sigma_a = 0$) and of leakage ($\vec{J} = 0$), the number of n's crossing each energy is Q_f
  → No accumulation of n's at energy E

NEUTRON FLUX IN HYDROGEN

- What we really want is solve for $\phi(E)$: $$q(E) = \int_E^{E/\alpha} \frac{E-\alpha E'}{(1-\alpha)E'}\,\Sigma_s(E')\phi(E')dE'$$
- First in hydrogen while neglecting absorption
- Taking the derivative with respect to E of the second fundamental slowing down equation (see slide 16):
$$\frac{\partial q}{\partial E} = \int_E^{\infty} \frac{\Sigma_s(E')\phi(E')}{E'}\,dE' - \Sigma_s(E)\phi(E) = 0$$
- Assuming that $\Sigma_s(E)$ is constant (good approximation < 10keV) :
$$\phi(E) = \int_E^{\infty} \frac{\phi(E')}{E'}\,dE'$$
- Differentiating both sides with respect to E: $$\frac{d\phi(E)}{dE} = -\frac{\phi(E)}{E}$$
- Thus, $\phi(E) = \frac{C}{E}$ with $C = {}^{Q_f}/_{\Sigma_s}$

→ the slowing-down spectrum in hydrogen neglecting absorption is ~ 1/ E

- What about in Graphite, e.g. for A> 1?

Slowing down equations

Neutron flux in non-absorbing medium, A > 1

- What we really want is determining $\phi(E)$ e.g. solve: $q(E) = \int_E^{E/\alpha} \frac{E-\alpha E'}{(1-\alpha)E'} \Sigma_s(E')\phi(E')dE'$

- The solution $\phi(E) = \frac{C}{E\,\Sigma_s(E)}$ (C constant) satisfies the constraint $q(E)$ = cst, for $E < E_{min}$

$$q(\vec{r}, E) = \int_E^{E/\alpha} \frac{E - \alpha E'}{(1-\alpha)E'} \Sigma_s(E')\,\phi(E')\,\mathrm{d}E'$$

- Thus, $\phi(E) = \dfrac{Q_f}{E\,\xi\,\Sigma_s(E)}$

  → Since Σ_s ~ constant in practice, the slowing-down spectrum is also ~ 1/ E

- This is one possible solution of the problem. Obtaining the general form is much more involved (solution of Placzek transient)

- The 1/E solution represents the asymptotic flux, e.g. the flux after a large number of collisions. In practice it is a good estimator of the flux in non-absorbing media

LETHARGY

- The number of scattering events with H increases with decreasing energy:

$$\Sigma_s \, \phi(E) = \frac{Q_f}{E}$$

- E.g. # of collisions per unit E at 1eV is 10^6 times higher than at 1MeV

  ➞ We introduce instead of E a new unit in which the collision density ($\Sigma_s \phi$) changes much less

- Neutron lethargy: $u = \ln \dfrac{E_0}{E}$

$$E = E_0 \exp(-u)$$

- E_0 is an arbitrary energy usually the energy of the fastest neutrons

- Neutrons are born with u = 0. When they slow down, their lethargy increases.

- $\Sigma_s\phi(u)$ and $\Sigma_s\phi(E)$ are different functions, but collisions occurring in d*u* are the same collisions that occur in d*E*

$$\Sigma_s\phi(u)\,\mathrm{d}u = -\Sigma_s\phi(E)\,\mathrm{d}E$$

- $u = \ln\frac{E_0}{E} \;\Rightarrow\; \mathrm{d}u = -\frac{\mathrm{d}E}{E} \;\Rightarrow\; \phi(u) = \phi(E)E$

- For *hydrogen*: $\phi(E) = \frac{Q_f}{E\,\Sigma_s} \;\Rightarrow\; \phi(u) = \frac{Q_f}{\Sigma_s}$

- Assuming Σ_s=cst, energy-dependent flux per unit lethargy is *constant* for H.

- With every collision the energy of a neutron decreases and the lethargy increases by

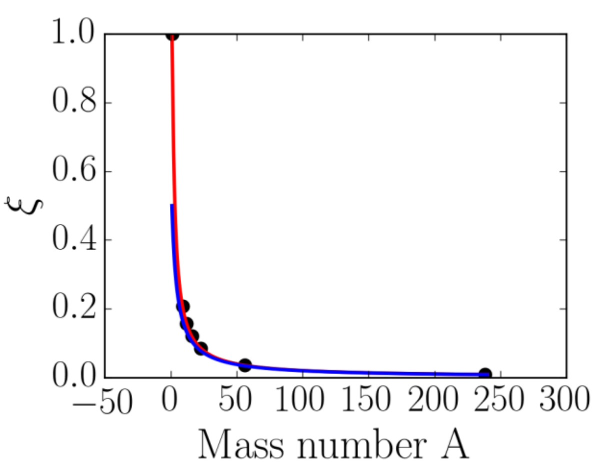
$$\Delta u = \ln\frac{E_0}{E'} - \ln\frac{E_0}{E} = \ln\frac{E}{E'}$$

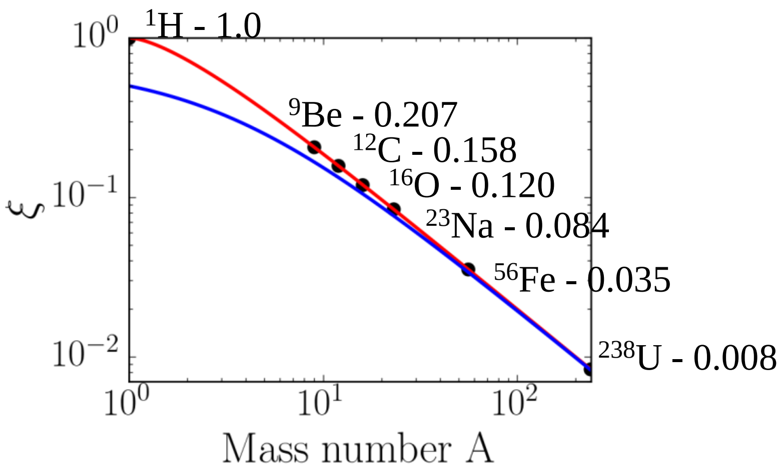
- *Average* increase of lethargy per collision can be calculated from the integral

$$\xi = \overline{\Delta u} = \int_{\alpha E}^{E} \ln\frac{E}{E'} P_1(E \to E')\,\mathrm{d}E' = \frac{1}{E(1-\alpha)}\int_{\alpha E}^{E}\ln\frac{E}{E'}\,\mathrm{d}E'$$

- Making the substitution: $x = \frac{E'}{E} \quad \Rightarrow \quad \xi = 1 + \frac{\alpha\ln\alpha}{1-\alpha}$

- Using $\alpha = \left(\frac{A-1}{A+1}\right)^2, \quad \xi = 1 - \frac{(A-1)^2}{2A}\ln\frac{A+1}{A-1} \quad \simeq \frac{2}{A+3}$ if $A > 10$





SLOWING DOWN IN MIXTURES OF NUCLIDES

- In case of mixture of the scattering nuclei (e.g. H_2O), flux

$$\phi(E) = \frac{S}{\bar{\xi}(E)\, E\, \Sigma_s(E)}$$

- where average increase of lethargy per collision for a mixture of nuclides

$$\bar{\xi}(E) = \frac{\sum_i \xi_i \Sigma_{s,i}(E)}{\Sigma_s(E)}$$

→ $\xi\Sigma_s$ is called ***slowing-down power***

→ $\frac{\xi\Sigma_s}{\Sigma_a}$ is called ***moderating ratio***

Material	ζ	Number of collisions to thermalize	Slowing Down Power [cm^{-1}]	Moderating Ratio
H_2O	0.927	19	1.425	62
D_2O	0.51	35	0.177	4830
Helium	0.427	42	$9\ 10^{-5}$	51
Berylium	0.207	86	0.154	126
Boron	0.171	105	0.092	0.00086
Carbon	0.158	114	0.083	216

- Till now: infinite, homogeneous media → Φ uniform (same for all positions)

- In practice, one has a reactor of finite dimensions, non-homogeneous
  → There is a relationship between $\Phi(E)$ and the distance from the source
  → Numerical approach (multigroup theory) allows accurate treatment of neutron behavior (see Lecture #10)

- A simplified treatment allows one to obtain analytical solutions (Fermi's theory)

- Corresponding hypotheses:
  → λ_t does not vary strongly with energy
  → ξ is small (slowing down almost continuous)
  → $\Sigma_a \sim 0$
  → Neutron spectrum not affected by differential leakage (greater leakage for fast n's)
  → Diffusion theory is valid

EPFL

- One considers the neutron balance in a bare homogeneous reactor, in the volume $\mathrm{d}V(\vec{r})$

$$q(\vec{r}, E+\mathrm{d}E) - q(\vec{r}, E) = \frac{\partial q(\vec{r}, E)}{\partial E}\mathrm{d}E$$

- The change is due to leakage…

$$-D(E)\nabla^2\phi(\vec{r}, E) = \frac{\partial q(\vec{r}, E)}{\partial E}$$

- In absence of absorptions, one can show that

$$\phi(\vec{r}, E) = \frac{q(\vec{r}, E)}{\xi\,\Sigma_s(E)E}$$

- Thus, $-\dfrac{D(E)}{\xi\,\Sigma_s(E)E}\nabla^2 q(\vec{r}, E) = \dfrac{\partial q(\vec{r}, E)}{\partial E} \quad \Rightarrow \quad \nabla^2 q(\vec{r}, E) = -\dfrac{\xi\,\Sigma_s(E)E}{D(E)}\dfrac{\partial q(\vec{r}, E)}{\partial E}$

- Defining "Fermi Age" corresponding to energy E by $\tau(E) = \displaystyle\int_E^{E_0} \frac{D(E')}{\xi\Sigma_s(E')}\frac{\mathrm{d}E'}{E'}$

$$\rightarrow \quad \frac{\partial\tau(E)}{\partial E} = -\frac{D(E')}{\xi\Sigma_s(E')}\frac{\mathrm{d}E'}{E'} \quad \Rightarrow \quad \nabla^2 q(\vec{r}, \tau) = \frac{\partial q}{\partial \tau}$$

(***Age equation***)

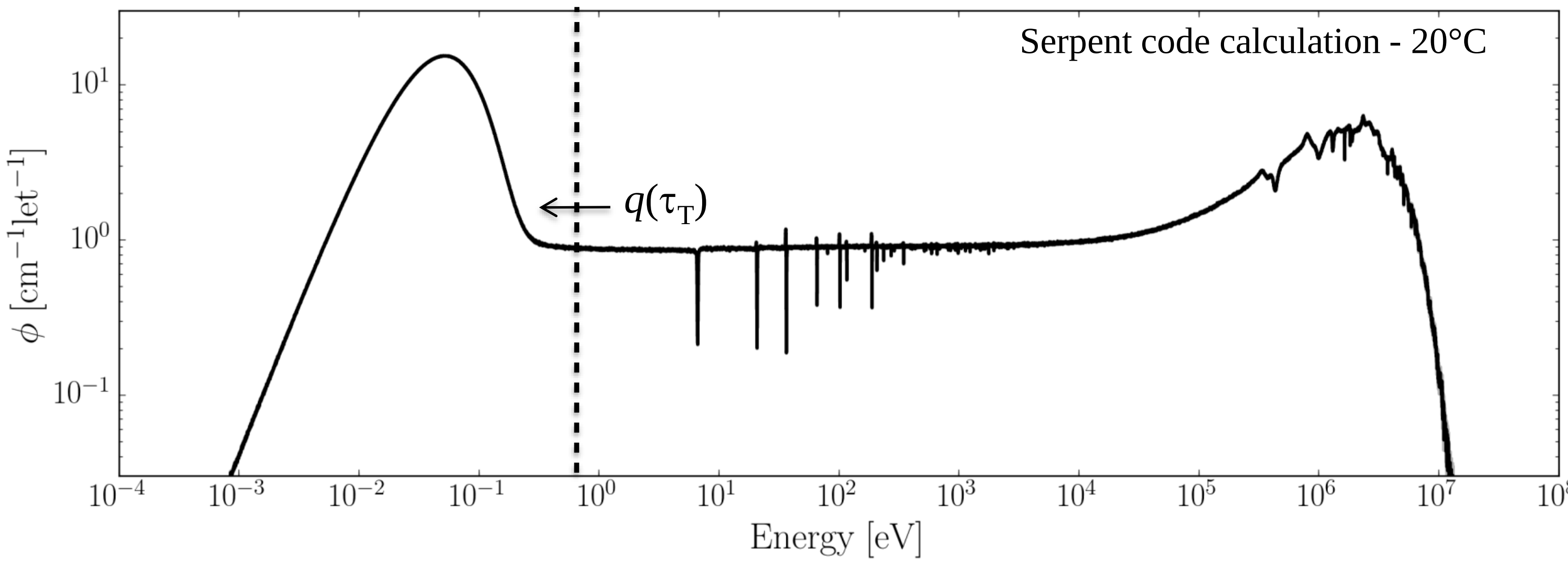
- Looks like the heat conduction equation, except that τ has the dimensions of area, not of time

- Thermal and fast fluxes are just integrals of energy-dependent flux over thermal and above-thermal regions (resonance+fast):

$$\phi_\mathrm{T} = \int_0^{\sim 5k\,T} \phi(E)\,\mathrm{d}E \qquad \phi_\mathrm{F} = \int_{\sim 5k\,T}^{\infty} \phi(E)\,\mathrm{d}E$$

- Slowing-down density $q(\tau_T)$ is a source of thermal neutrons

Neutron spectrum in homogeneous (^{235}U / ^{238}U / ^{1}H / ^{16}O) - (0.05 / 0.95 / 100 / 50) mixture



EPFL

- Steady-state diffusion equation for thermal neutrons:

$$D_{\mathrm{T}} \nabla^2 \phi_{\mathrm{T}}(\vec{r}) - \Sigma_{\mathrm{T}} \phi_{\mathrm{T}}(\vec{r}) + q(\vec{r}, \tau_{\mathrm{T}}) = 0$$

- For ***one-zone*** homogeneous reactor the thermal slowing-down density (a source of thermal n's) can be found using Fermi age equation:

$$\nabla^2 q(\vec{r}, \tau) = \frac{\partial q(\vec{r}, \tau)}{\partial \tau}$$

Considering a ***one-zone*** homogeneous infinite (in *z* and *y* directions) slab reactor of extrapolated thickness *a*, solving for the thermal flux:

- Diffusion equation: $$\frac{\mathrm{d}^2\phi_\mathrm{T}(x)}{\mathrm{d}x^2} - \frac{\phi_\mathrm{T}(x)}{L^2} = -\frac{q(x,\tau_\mathrm{T})}{D}$$

- Age equation: $$\frac{\partial^2 q(x,\tau)}{\partial x^2} = \frac{\partial q(x,\tau)}{\partial \tau}$$

- Source condition (for simplicity we assume fission neutrons as monoenergetic): $$q(x,0) = \Sigma_\mathrm{T}\phi_\mathrm{T}(x)\, f\,\eta_\mathrm{T}\,\epsilon = \Sigma_\mathrm{T}\phi_\mathrm{T}(x)\frac{k_\infty}{p}$$

- Boundary and symmetry conditions:
$$q(\pm a/2,\tau) = 0$$
$$q(-x,\tau) = q(x,\tau)$$
$$\phi_\mathrm{T}(\pm a/2) = 0$$
$$\phi_\mathrm{T}(-x) = \phi_\mathrm{T}(x)$$

Recall the slab eigenvalues and eigenfunctions (lecture 7):

- Eigenfunctions $\phi_n(x) = \cos(\frac{n\pi x}{a}) = \cos(B_n x)$
- Eigenvalues $B_n = \frac{n\pi}{a}, \quad \text{where } n = 1, 3, 5, \ldots$
- Then thermal flux can be decomposed as:

$$\phi_{\mathrm{T}}(x) = \sum_{n \text{ odd}} A_n \cos(B_n x)$$

- Using the Age equation and the q(x,0) boundary condition, the slowing down source can be expressed as:

$$q(x,\tau) = \frac{\Sigma_T k_\infty}{p} \sum_{n\ odd} A_n \exp(-B_n^2 \tau) \cos(B_n x)$$

- It can be shown (see e.g. Section 9.2 of Lamarsh) that, when considering solutions of the time-dependent diffusion equation with both external and fission sources, all higher harmonics quickly die out after switching the external source ***off***.

- We will consider further only the fundamental solution:

$$\begin{aligned}\phi_{\mathrm{T}}(x) &= A\cos(Bx)\\ q(x,\tau_{\mathrm{T}}) &= \Sigma_{\mathrm{T}} A\cos(Bx) k_{\infty}\exp(-B^2\tau_{\mathrm{T}})\\ &= \Sigma_{\mathrm{T}}\phi_{\mathrm{T}}(x)\mathrm{k}_{\infty}\exp(-\mathrm{B}^2\tau_{\mathrm{T}})\end{aligned}$$

$$\text{where } B = \frac{\pi}{a}$$

- Inserting these solutions into the original diffusion equation:

$$\frac{\mathrm{d}^2\phi_\mathrm{T}(x)}{\mathrm{d}x^2} - \frac{\phi_\mathrm{T}(x)}{L^2} = -\frac{q(x,\tau_\mathrm{T})}{D}$$

$$\Rightarrow -B^2\phi_\mathrm{T}(x) + \frac{k_\infty \exp(-B^2\tau_\mathrm{T}) - 1}{L^2}\phi_\mathrm{T}(x) = 0$$

$$\Rightarrow B^2 = \frac{k_\infty \exp(-B^2\tau_\mathrm{T}) - 1}{L^2}$$

- Therefore:

$$\frac{k_\infty \exp(-B^2\tau_\mathrm{T})}{1 + B^2L^2} = 1 \qquad \text{and} \qquad \frac{\mathrm{d}^2\phi_\mathrm{T}(x)}{\mathrm{d}x^2} + B^2\phi_T(x) = 0$$

This is the ***criticality condition***

This is the ***reactor equation***

- A reactor is considered large when the reactor size r is much bigger than a slowing down length $\sqrt{\tau_{\mathrm{T}}}$

- Since $B^2 \sim 1/r^2$, $B^2\tau_{\mathrm{T}} << 1$ and $\exp(B^2\tau_{\mathrm{T}}) \approx 1 + B^2\tau_{\mathrm{T}}$

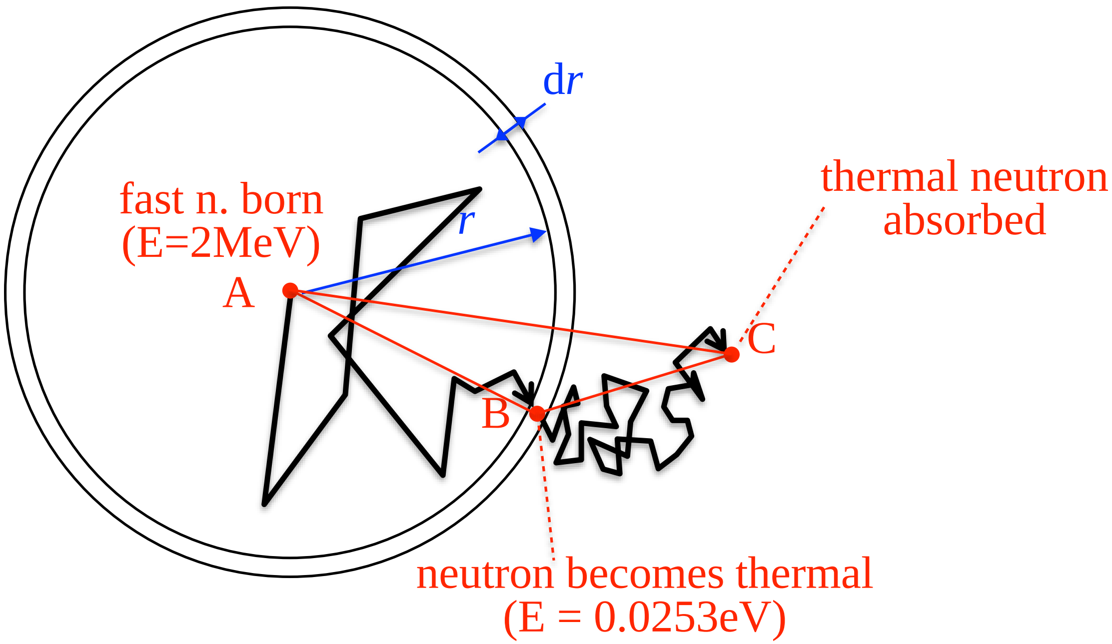
- The critical equation:

$$1 = \frac{k_\infty \exp(-B^2\tau_{\mathrm{T}})}{1 + B^2L^2} \simeq \frac{k_\infty}{(1 + B^2\tau_{\mathrm{T}})(1 + B^2L^2)} \simeq \frac{k_\infty}{1 + B^2(\tau_{\mathrm{T}} + L^2)}$$

- The thermal migration area: $M^2 = \tau_{\mathrm{T}} + L^2$

$$\frac{k_\infty}{1 + B^2M^2} = 1$$

- This is a critical equation of *one-group modified* theory



Age to thermal : $\tau_T = \frac{1}{6}\overline{|AB|}^2$

Diffusion area : $L^2 = \frac{1}{6}\overline{|BC|}^2$

Migration area : $M^2 = \tau_T + L^2$

SUMMARY

- A thermal reactor has n's between ~ 2 MeV (fission n's) and ~ 0.01 eV (thermal n's). Moderator is material which is used to slow down (moderate) n's
- Most important slowing-down mechanism: elastic scattering by moderator nuclei
- Energy loss in a single elastic collision is proportional to scattering angle
- The maximum fractional energy loss in a single elastic collision $(1 - \alpha)$ decreases with increasing mass of the struck nucleus
- Lethargy is another energy unit which shows the change of neutron's energy in logarithmic scale. A change of lethargy per collision $\xi \sim 2/(A+3)$
- Slowing down in hydrogen (neglecting absorption) results in a "one-over-E" spectrum

- Fermi proposed a simple differential equation to describe slowing down density as a function of space and a Fermi age similar to heat conduction equation. Allows to obtain analytical solutions for bare homogeneous reactors.

- Fermi age (τ_T — age to thermal) is a parameter [cm^2] describing how far neutrons can travel from an emission point while slowing down to thermal energy.