Modelling and design of experiments

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Fall 2024

1.1.1 Organization of the course Fall 2024

Wednesday schedule

- **Lecture**	9h15-11h	(class room+zoom)
- **Exercises**	11h15-12h	(class room)
- **Project**	14h15-15h	(class room, MA students only)

Evaluation

- PhD students : report on your own case by group
- MA students : project presentation + questions of theory

1.1.2 Expected work

- Active participation during the lecture
- Conscious work on the exercises (statistical and computing aspects)
- Fill in the gaps
- 4 ects $\approx$ 60hrs of personal work : 4hrs/week

1.1.3 Resources

- Moodle page with the slides and the exercises
- Textbook
- Internet and library

1.1.4 References

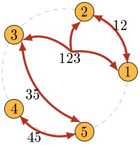
- Box, G., et al (2000), Statistics for experimenters, Wiley.
- Montgommery, D. (2009), Design and analysis of experiments, Wiley.
- Ryan,Th. (2007), Modern Experimental design, Wiley.
- Saltelli,A. (2000), Sensitivity analysis, Wiley.
- Lawson, J. (2014), Design and Analysis of Experiments with R, CRC press.

1.1.5 Softwares

- Matlab → the standard tool of the course
- Python → implies more personal work
- R-Markdown (R-Studio) → implies more personal work, but a very interesting alternative
- Excel → slower and a lot less powerful

1.2.1 Why learning DOE ?

- Nature answers questions in a very narrow way
- It is then key to question it with method
- DOE offer method to sharpen your experimental endeavor :
  - multifactorial approach
  - noise reduction strategy
  - taking interactions into account



Course chapters

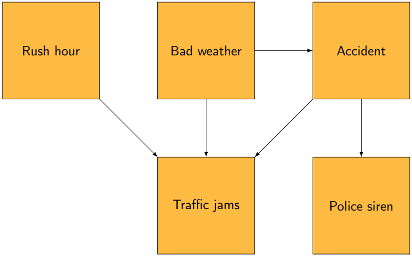
1. Introduction
2. Modeling
3. ANOVA
4. Factorial based designs
5. Response surface designs
6. Mixture designs
7. Qualitative factors

1.2.2 Pedagogical objectives of chapter 1

- Having a general orientation on the course
- Understand the origin of DOE
- Remember a few basic elements of statistics and data analysis
- Learning how to draw a mind map for a case
- Learning how to draw a causal model from data
- Training the basic operations of data analysis

1.2.3 Causes, effects and contingencies

- Scientific experiment aims to determine relations between causes and effects
- Necessity to deal with confusion factors (see later)
- Necessity to deal with contingencies such as delays, costs, batches, security, ...



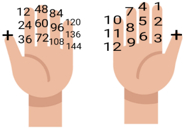
1.2.4 Nonlinearity and Cause-Effect Relationships

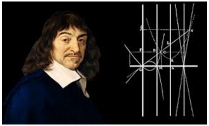
In nonlinear systems, the cause-effect relationship can be more complex due to several factors :

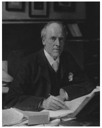
- **Nonlinearity in Response :**
  In linear systems, the response to an input is proportional. In nonlinear systems, the response can be disproportionate (e.g., doubling input might quadruple output).
- **Threshold Effects :**
  Nonlinear systems often have thresholds where behavior changes dramatically, making outcomes less predictable.
- **Feedback Loops :**
  Nonlinear systems may include feedback loops that amplify or dampen effects, complicating predictions.
- **Multiple Equilibria :**
  Multiple equilibrium points can lead to different outcomes for the same input depending on initial conditions.
- **Complex Interactions :**
  Nonlinear systems can exhibit chaos, where small changes in initial conditions lead to vastly different outcomes.

1.2.5 A (very) brief time line of statistics

- Sumer (III millennium BC)
  - Livestock control (list, coding)
  - Prediction of sunrises and sunsets, tides and floods
- Aristotle (384-322 BC)
  - Things that change all the time can not be the objects of science
- René Descartes (1596-1650), Pierre de Fermat (1607-1665) et Blaise Pascal (1623-1662)
  - Theory of probability
- Thomas Bayes (1702-1761) et Pierre Simon de Laplace (1749-1827)
  - Conditional probability, prior vs posterior information
- Francis Galton (1822-1911), Karl Pearson (1857-1936) and Ronald Fisher (1890-1962)
  - Genetics, eugenics, correlations







1.2.6 Sir Ronald Fisher (1890-1962)

- Agronomic station of Rothamsted, 1919
- Too frequent non-conclusive research
- Necessity of a collaboration between statisticians and experimenters
- Invention of ANOVA and DOE
- Statistical Methods for Research Workers (1925) (∼ *Newton's Principia*)
- The Design of Experiments (1935)
- Founder of the neo-Darwinism and of the modern genetics

Pause

1.3.1 Introduction to Data

- **Data** : Raw facts and figures collected through observations or measurements.
- **Types of Data** :
  - *Quantitative Data* : Numerical, can be discrete or continuous.
  - *Qualitative Data* : Categorical, can be nominal or ordinal.
- **Examples** : Test scores, survey responses, temperature readings.

1.3.2 Situations Based on Types of Data

Type of X	**Type of Y**	**Situation Description**
Qualitative	Qualitative	Comparing frequencies between groups
Qualitative	Quantitative	Comparing means or medians between groups
Quantitative	Qualitative	Logistic regression
Quantitative	Quantitative	Regression

1.3.3 Introduction to Metadata

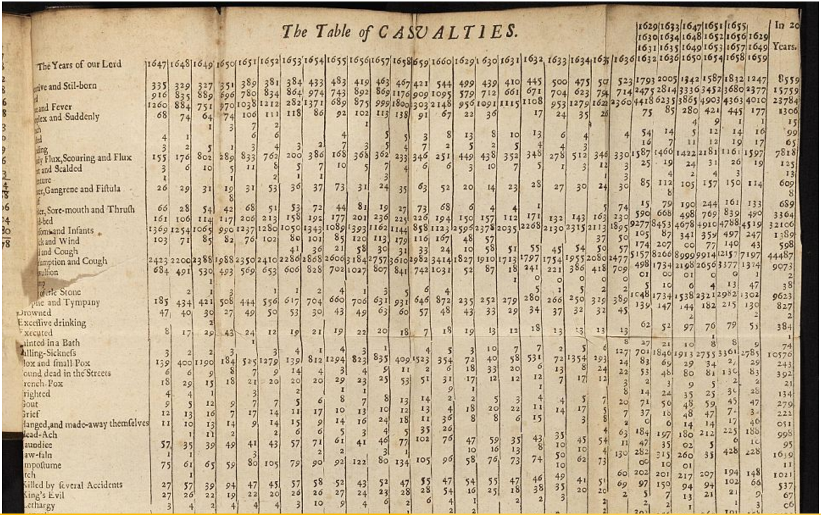
- **Metadata** : Data about data ; provides context and additional information.
- **Purpose** :
  - Helps in understanding the context, quality, and limitations of the data.
  - Supports data management and interpretation.
- **Examples** : Date of data collection, methodology, units of measurement, source.

1.3.4 Importance and Relationship

- **Importance of Data** :
  - Understanding data types helps in choosing appropriate statistical methods.
  - Ensures accurate analysis.
- **Importance of Metadata** :
  - Enhances data value by providing essential interpretive details.
  - Allows for better data management and replication.
- **Relationship** :
  - Metadata provides the context needed to interpret and analyze data effectively.
  - Whenever it is possible, metadata must be integrated in the data file
  - **Laboratory notebooks** are also key to keep the coherence of the data and metadata

Graunt’s book 1663

Natural and Political Observations Made upon the Bills of Mortality



1.3.5 Categorization : Titanic passengers

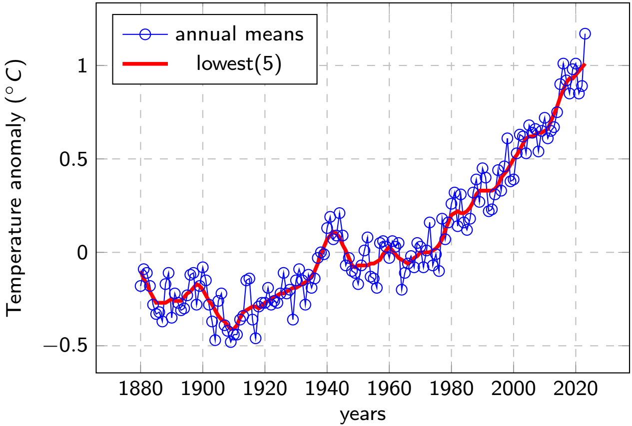
		Dead		Survivors	
Age	Class	M	F	M	F
Children	1st	0	0	5	1
	2nd	0	0	11	13
	3rd	35	17	13	14
	Crew	0	0	0	0
Adults	1st	118	4	57	140
	2nd	154	13	14	80
	3rd	387	89	75	76
	Crew	670	3	192	20

"*[...] Those methods deal comprehensively with entire species, and with entire groups of influences, just as if they were single entities, and express the relations between them in an equally compendious manner. They commence by marshalling the values in order of magnitude from the smallest up to the largest, thereby converting a mob into an orderly array, which like a regiment thenceforth becomes a tactical unit.* ", F. Galton, Biometrika, Volume 1, Issue 1, October 1901

1.3.6 Data : Earth Surface Temperature (NASA)

year	crude	lowest(5)	year	crude	lowest(5)	year	crude	lowest(5)
1901	-0.15	-0.23	1951	-0.07	-0.07	2001	0.53	0.52
1902	-0.28	-0.26	1952	0.01	-0.07	2002	0.63	0.55
1903	-0.37	-0.28	1953	0.08	-0.07	2003	0.62	0.58
1904	-0.47	-0.31	1954	-0.13	-0.06	2004	0.53	0.61
1905	-0.26	-0.34	1955	-0.14	-0.06	2005	0.68	0.62
1906	-0.22	-0.36	1956	-0.19	-0.05	2006	0.64	0.62
1907	-0.39	-0.37	1957	0.05	-0.04	2007	0.66	0.63
1908	-0.42	-0.39	1958	0.06	-0.01	2008	0.54	0.64
1909	-0.48	-0.41	1959	0.03	0.01	2009	0.65	0.64
1910	-0.44	-0.41	1960	-0.03	0.03	2010	0.72	0.65
1911	-0.44	-0.39	1961	0.06	0.01	2011	0.61	0.66
1912	-0.36	-0.35	1962	0.03	-0.01	2012	0.65	0.7
1913	-0.34	-0.32	1963	0.05	-0.03	2013	0.67	0.74
1914	-0.15	-0.31	1964	-0.2	-0.04	2014	0.75	0.78
1915	-0.14	-0.3	1965	-0.11	-0.05	2015	0.9	0.83
1916	-0.36	-0.29	1966	-0.06	-0.06	2016	1.01	0.87
1917	-0.46	-0.29	1967	-0.02	-0.05	2017	0.92	0.91
1918	-0.29	-0.3	1968	-0.08	-0.03	2018	0.85	0.93
1919	-0.27	-0.29	1969	0.05	-0.02	2019	0.98	0.93
1920	-0.27	-0.27	1970	0.03	0	2020	1.01	0.95
1921	-0.19	-0.26	1971	-0.08	0	2021	0.85	0.97
1922	-0.28	-0.25	1972	0.01	0	2022	0.89	0.99
1923	-0.26	-0.24	1973	0.16	0	2023	1.17	1.01
1924	-0.27	-0.23	1974	-0.07	0.01			
1925	-0.22	-0.22	1975	-0.01	0.02			
...	...	...	...	...	...			

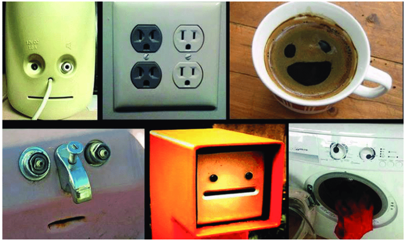
1.3.7 A picture is worth a thousand words



1.3.8 Importance and Risks

- **Importance :**
  - Categorizing data or creating visual representations helps in identifying patterns, trends, and relationships.
  - Graphs make complex data more accessible and easier to interpret.
  - Visual summaries provide a quick overview of data distribution and key features.
- **Risks of Bias :**
  - Categorization can oversimplify or obscure important variations within data.
  - Choice of graph type, scale, or binning can mislead or distort the true nature of data.
  - Visual representations might exaggerate or minimize effects, leading to misinterpretations.
- **Key Takeaway :**
  - While categorization and graphing are powerful tools, analysts must remain mindful of biases that can affect data interpretation.

1.3.9 Pareidolia



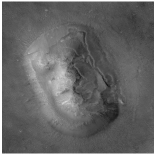
Our brain has the tendency to see motives in random sets as well as a tendency to make a script, which means to tell a story, and give a meaning to an image. Statistics offers tools, like statistical tests, to pass the perception.

1.3.10 Cydonia Mensae

- The « Mars' face » put in evidence the influence of the resolution of a numerical image over the interpretations than is given of it.
- The principle of parsimony of the hypotheses : it is less costly in hypotheses to consider that our brain looks for and finds a meaning to this picture, recognizing a human face in a picture, than to consider the existence of an extraterrestrial civilization who would have built a huge construction representing a human face.

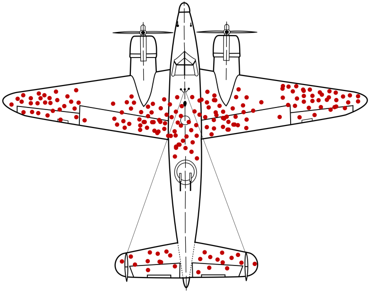
https://fr.wikipedia.org/wiki/Cydonia_Mensae

Viking (1971)



Mars global surveyor (2001)

1.3.11 Survivorship bias



The damaged portions of returning planes show locations where they can sustain damage and still return home ; those hit in other places presumably do not survive. (Image shows hypothetical data.) The error was to consider that the planes are *more probably* hit in those places and then reinforce them.

1.3.12 Pitfalls in data analysis

Cognitive biases

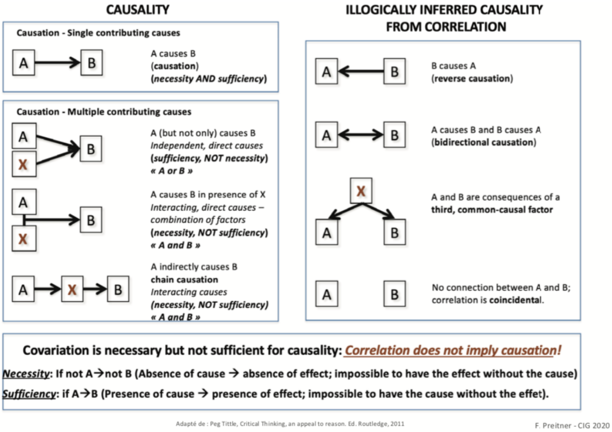
- illusions
- of attention
- of memory
- of judgment
- of reasoning

Examples

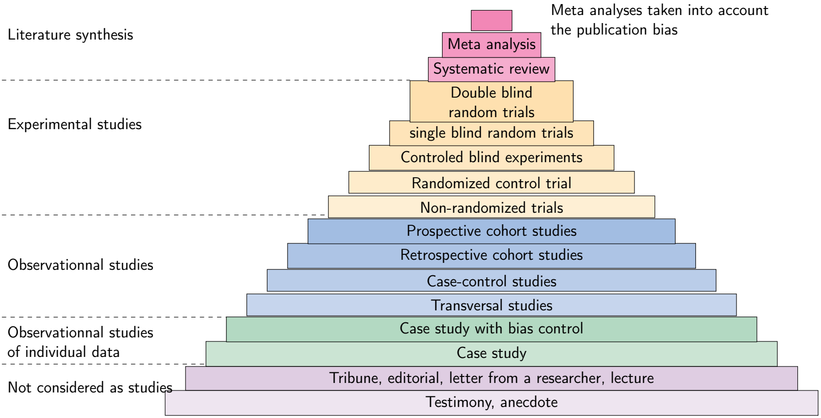
- Pareidolia
- Confusion correlation/causation
- Bias of confirmation (Cherry picking)
- Bias of belief
- Error of attribution
- Barnum effect (experience of Forer)

https://en.wikipedia.org/wiki/List_of_cognitive_biases

1.3.13 Causality



1.3.14 Scientific consensus



1.3.15 Falsifiability (Popper)

- A statement is said refutable (falsifiable) if and only if it can be logically contradicts by an empirical test. ... more precisely if and only if it exists a possible observation statement (true or false) that would logically contradict the theory.
- To be accepted as scientific a statement, a theory must be falsifiable.

1.3.16 Simpson paradox

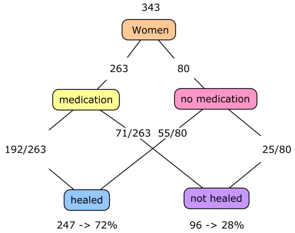
The **fraction of healing** of 700 patients with and without the use of a given drug. 350 individuals with the medication (group "with"), 350 without the medication (group "without").

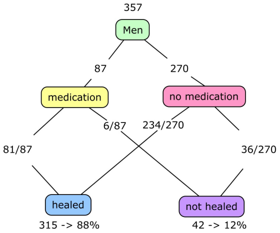
	With	**Without**
Men	81/87	234/270
Women	192/263	55/80
All	273/350	289/350

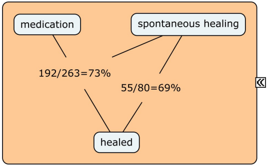
	With	**Without**
Men	93%	87%
Women	72%	69%
All	78%	83%

Is the drug to be proposed or not ?

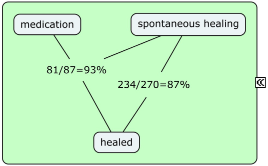
1.3.17 Cause analysis







delta=4%



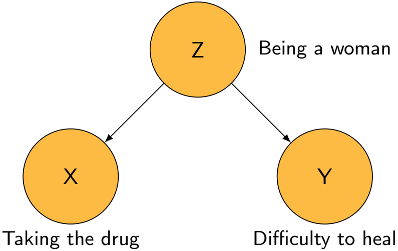
delta= 6%

1.3.18 Analysis

- If **taking the drug** appears less effective than **doing nothing**, it is because if we randomly choose a person who has taken the drug, the probability is high that this person is a woman and therefore that his recovery is less easy than a person chosen at random from those who did not use the drug, in which case the probability is greater that it is a man and that he recovered more easily.
- In the example, **being a woman** is a common cause **to take the medicine** and **to heal less easily**.
- In this case, the aggregated data does not allow to really test the effectiveness of the drug.
- To test the efficacy of the drug **a homogeneous group** is needed so that the difference in cure rate is attributable to the drug alone and not to the effect of estrogen.

1.3.19 Confounding factor

A confounding factor, (or extraneous determinant or lurking variable) is a variable that influences both the dependent variable and independent variable, causing a spurious association. Confounding is a causal concept, and as such, cannot be described in terms of correlations or associations



Pause

1.3.20 Data about scientific publications

1	Country	Code	Region	Income	Level	2 000	2 018	pop2000	pop2018
2	Afghanistan	AFG	South Asia	Low income	1	4	112	20 779 957	37 171 922
3	Angola	AGO	Sub-Saharan Africa	Lower middle income	2	7	30	16 395 477	30 809 787
4	Albania	ALB	Europe & Central Asia	Upper middle income	3	22	180	3 089 027	2 866 376
5	Andorra	AND	Europe & Central Asia	High income	4	-	4	65 390	77 008
6	United Arab Emirates	ARE	Middle East & North Africa	High income	4	330	3 145	3 134 067	9 630 966
7	Argentina	ARG	Latin America & Caribbean	Upper middle income	3	4 386	8 811	36 870 796	44 494 502
8	Armenia	ARM	Europe & Central Asia	Upper middle income	3	346	521	3 069 597	2 951 741
9	Antigua and Barbuda	ATG	Latin America & Caribbean	High income	4	0	6	76 007	96 282
10	Australia	AUS	East Asia & Pacific	High income	4	23 276	53 610	19 153 000	24 982 688
11	Austria	AUT	Europe & Central Asia	High income	4	6 577	12 362	8 011 566	8 840 521
12	Azerbaijan	AZE	Europe & Central Asia	Upper middle income	3	155	761	8 048 600	9 939 771
13	Burundi	BDI	Sub-Saharan Africa	Low income	1	2	21	6 378 871	11 175 379
14	Belgium	BEL	Europe & Central Asia	High income	4	9 723	15 688	10 251 250	11 427 054
15	Benin	BEN	Sub-Saharan Africa	Lower middle income	2	43	228	6 865 946	11 485 035
16	Burkina Faso	BFA	Sub-Saharan Africa	Low income	1	48	252	11 607 951	19 751 466
17	Bangladesh	BGD	South Asia	Lower middle income	2	440	3 135	127 657 862	161 376 713
18	Bulgaria	BGR	Europe & Central Asia	Upper middle income	3	1 653	3 311	8 170 172	7 025 037
19	Bahrain	BHR	Middle East & North Africa	High income	4	83	322	664 610	1 569 440
20	Bahamas, The	BHS	Latin America & Caribbean	High income	4	2	20	298 045	385 635
21	Bosnia and Herzegovina	BIH	Europe & Central Asia	Upper middle income	3	75	704	3 751 176	3 323 929
22	Belarus	BLR	Europe & Central Asia	Upper middle income	3	1 170	1 180	9 979 610	9 483 499
23	Belize	BLZ	Latin America & Caribbean	Lower middle income	2	1	9	247 310	383 071
24	Bolivia	BOL	Latin America & Caribbean	Lower middle income	2	39	103	8 418 270	11 353 140
25	Brazil	BRA	Latin America & Caribbean	Upper middle income	3	12 783	60 148	174 790 339	209 469 320

https://data.worldbank.org/indicator/IP.JRN.ARTC.SC?end=2018&start=2000

1.3.21 Loading data

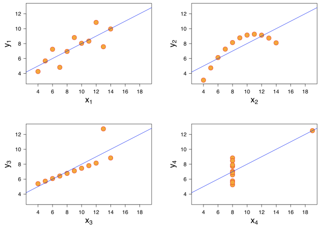
Matlab has specialized routines to load data in the workspace

MATLAB

- *X=0 :1 :20* creates the vector $X = [0, 1, 2, \ldots, 20]$
- *X=[. . .]* creates a vector with specific values
- *Y=repmat(X,h,w)* creates a matrix copying X, h times vertically w times horizontally
- *T = table(var1, ..., varN, Name, Value)* creates a table
- *T = readtable(FileName.xls','Sheet',SheetName)* loads a table from Excel

1.3.22 ANSCOMBE'S QUARTET

- Four data sets that have nearly identical simple descriptive statistics, yet have very different distributions and appear very different when graphed.
- Each dataset consists of eleven (x,y) points
- Constructed in 1973 by the statistician Francis Anscombe to demonstrate both the importance of graphing data when analyzing it
- Intended to counter the impression among statisticians that "numerical calculations are exact, but graphs are rough."

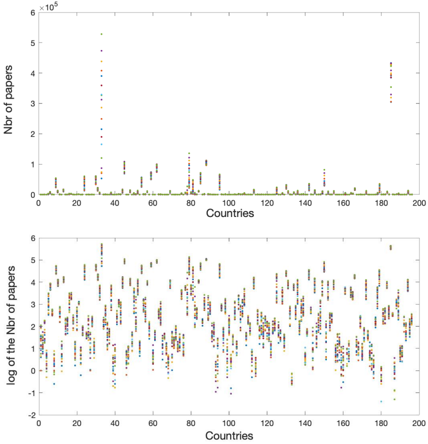


1.3.23 Visual analysis

- To detect patterns, aberrations, etc.
- The change of the metric is also interesting

Matlab

- *plot(x, y)*
  plot(x, y, LineSpec)
  plot(x_1, y_1, . . ., x_n, y_n)
  plot(. . .,Name,Value)
- *bar(x, y)*
  bar(. . ., Width)
  bar(. . ., Style)
  bar(. . ., Name, Value)



1.3.24 Sorting data

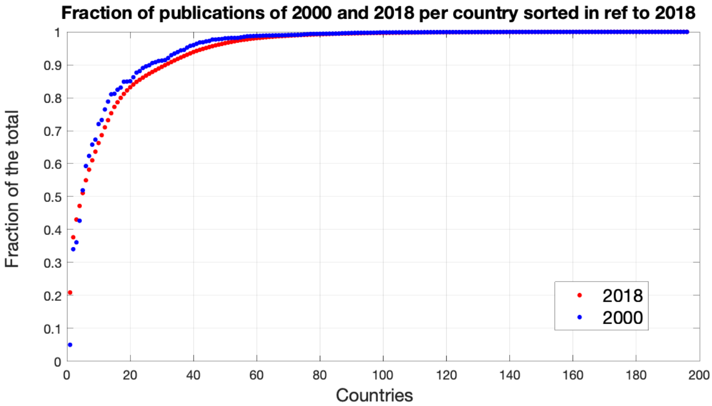
$$x_1 \ x_2 \ \ldots x_N \quad \Rightarrow \quad x_{(1)} \ < x_{(2)} \ < \cdots < x_{(N)}$$

MATLAB

- *[B,Index]=sort(A,dim,direction)*
- *[B,Index]=sortrows(A,col,direction)*
- *[tblB,Index]=sortrows(tblA,col,direction)*

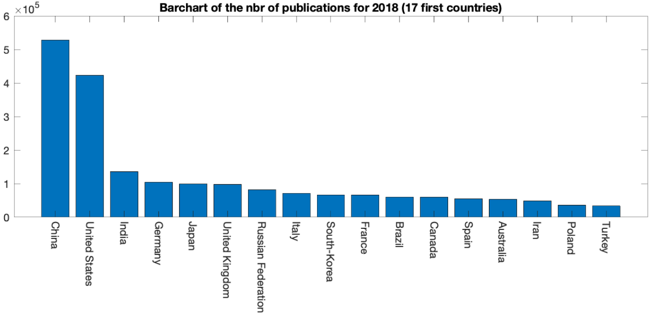
  A, B : matrices
  tblA, tblB : table
  dim : dimension to realize the sorting (1,2, . . .)
  col : column of reference for sorting
  direction : 'ascend' or 'descend'

1.3.25 Plot of the sorted data



1.3.26 Pareto principle

80% -20% principle : A maximum of things are related to a minimum of causes



1.3.27 Dealing with categories

Defining a column of a table as *categorical* allows you to perform some computation by categories

MATLAB

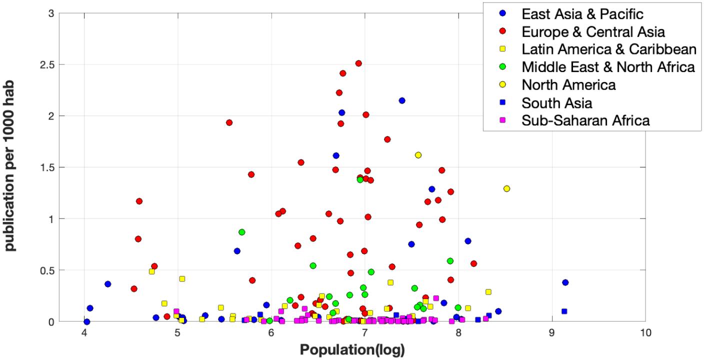
- *B=categorical(A)*

  tbl.var = categorical(tbl.var)
- *statarray = grpstats(tbl,group,stats)*

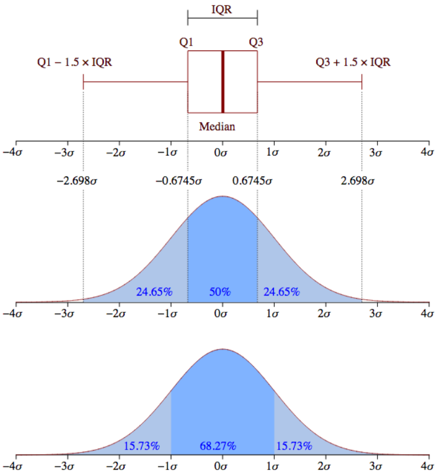
  stats=grpstats(tbl,'var1',{ 'min','max' },'tblVars','var2')
- *gscatter(x,y,group)*

  A,B : array or column of a table
  tbl : table
  group : the variable(s) of the groupe
  stats : statistics to compute such as min, max, etc.

1.3.28 Scatter plot

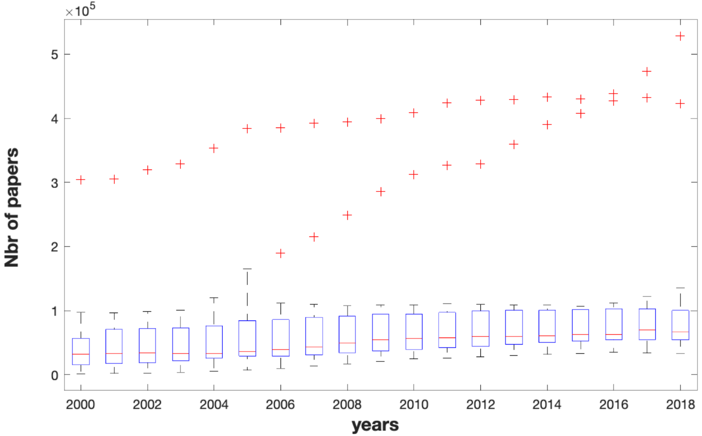


1.3.29 Box-plot (definition)



- *Outliers* are points placed at more than 1.5 IQR at the left of Q_1 or at the right of Q_3
- *Wiskers* are drawn at the minimum or maximum of the data points after the exclusion of the outliers

1.3.30 Box-plot



1.3.31 Location and range

$$x_{(1)}\ , x_{(2)}\ , \dots, x_{(N/4)}, \dots, x_{(N/2)}, \dots, x_{(3N/4)}, x_{(N-1)}, x_{(N)}$$

- Median
- Average
- Range
- Standard deviation
- Variance
- Quartile
- Percentile

MATLAB

M = median(A)
M = mean(A)
R = range(A)
S = std(A)
v = var(A)
Y = quantile(X,p)

1.3.31 Location and range

$$m_x = \frac{1}{N}\sum_{i=1}^{N} x_i$$

(N number of data point)

$$var(x) = s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - m)^2$$

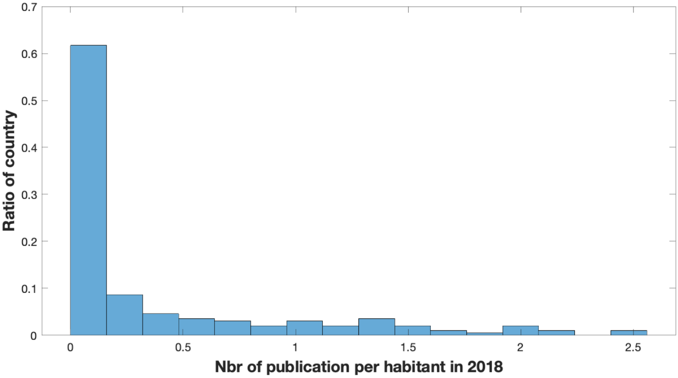
$$M = \begin{cases} \frac{1}{2}\left(x_{(N/2)} + x_{(N/2+1)}\right) \\ x_{((N+1)/2)} \end{cases}$$

$$s_x = \sqrt{s_x^2} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - m)^2}$$

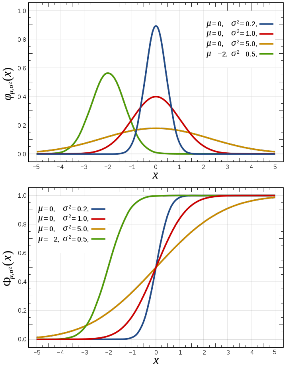
$$R = x_{(N)} - x_{(1)}$$

1.3.32 Quartiles and histogram

	Mean	Deviation	Variance	Q25	Median	Q75
2000	0.18	0.36	0.13	0.0025	0.018	0.11
2018	0.37	0.57	0.33	0.0128	0.08	0.48



1.3.33 Normal distribution $\mathrm{N}(\nu, \sigma)$



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The Normal distribution is characterized by its bell-shaped curve, which is symmetric around its mean. It describes how values of a variable are distributed, with most of the data clustering around the central mean μ and decreasing in frequency as they move further away.

The spread of the distribution is determined by its standard deviation, σ.

The Normal distribution is fundamental due to its natural occurrence in many real-world phenomena and its properties, such as the central limit theorem, which states that the sum of a large number of random variables tends to follow a normal distribution, regardless of the original distribution of the variables.

1.3.33 Normal distribution $\mathrm{N}(\nu, \sigma)$

MATLAB

- Random number generation
  X=randn(Ni,Nj)
- Probability density function
  p=pdf('Normal',x,mu,sigma)
- Cumulative density function
  p=cdf('Normal',x,mu,sigma)
- Inverse cumulative distribution function
  x=icdf('Normal',p,mu,sigma)

1.3.34 Data vs distributions

- Original data : $Y_i \sim N(\mu_i, \sigma_i)$ avec $0 \leqslant i \leqslant n$
- Linear : $a_j = \sum_i x_{ij} Y_i \sim N\left(\mu = \sum x_{ij}\mu_i,\ \sigma = \sqrt{\sum x_{ij}\sigma_i^2}\right)$
- Average : $\sqrt{n-1}\ \left(\frac{\bar{Y}-\mu}{s}\right) \sim T(\nu)$
- Quadratic function : $(a_j)^2 \sim \chi^2(w_j)$
- Quadratic function quotient : $\frac{(a_j)^2}{(a_i)^2} \sim F(w_j, w_i)$

1.3.35 Student's T_ν distribution and CI

- Published by William Gosset, 1908, (Guinness)

  If the observations X_i are IID [1] then $\left(\frac{\overline{X}-\mu}{s/\sqrt{n}}\right) \sim T_{n-1}$

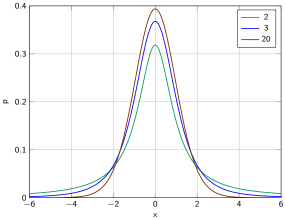
- Confidence interval theorem :

$$P\left(x \in \left[\bar{x} - t^{n-1}_{\alpha/2}\sqrt{\frac{S^2}{n}}\ ,\ \bar{x} + t^{n-1}_{\alpha/2}\sqrt{\frac{S^2}{n}}\right]\right) = 1 - \alpha$$

with t^ν_α the value of t at which $\int_0^t T_\nu(t')\,dt' = \alpha$

1. independent and identically distributed

1.36 The Student's T_ν distribution



	Parent dist.	sampling dist. for $\bar{y}$
Mean	η	η
Variance	σ^2	$\frac{\sigma^2}{n}$
Std dev.	σ	$\frac{\sigma}{\sqrt{n}}$
Form	$\sim$ any	more nearly Normal

The Student's t-distribution is used to estimate population parameters when the sample size is small and/or when the population variance is unknown.

The t-distribution is parameterized by degrees of freedom $\nu = n - p$ (n being the sample size and p, the number of parameters) ; as the degrees of freedom increase, the t-distribution approaches the normal distribution.

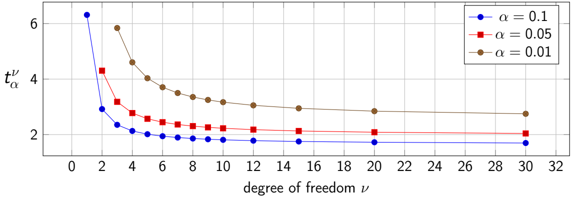
It is commonly used in hypothesis testing to determine if there is a significant difference between sample means.

1.3.37 Compute with the Student's distribution

MATLAB

- Generation of random number following t_ν
  X=trnd(nu,Ni,Nj)
- Probability density function of t_ν
  p=tpdf(x,nu)
- Cumulative density function of t_ν
  p=tcdf(x,nu) *p=tcdf(x,nu,'upper')*
- Inverse cumulative distribution function :
  x=tinv(p,nu)

1.3.38 Confidence and degrees of freedom



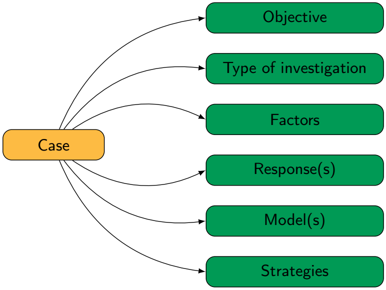
After about 10 degrees of freedom, the Student's t-distribution begins to closely resemble the normal distribution, and further increases in degrees of freedom result in only minor changes to the shape of the distribution. This means that, for practical purposes, you would need a much larger sample size (a change in the order of magnitude) to observe a significant reduction in the width of the confidence interval. Similarly, for confidence levels below 95%, the size of the confidence interval does not vary dramatically.

Therefore, using **10 degrees of freedom** and a **95% confidence** level is a reasonable standard for many statistical analyses, as it balances precision with practicality.

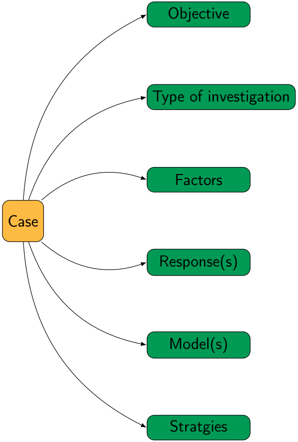
Pause

1.4.0 Mindmap of a case

- To do at the start of the project
- To maintain all along the project
- By hand or with a dedicated application
- Some available applications :
  - Freemind
  - iMindmap
  - Mindjet
  - MIRO



1.4.1 Building the mindmap



Usually, it concists in determinating the dominant factors, or the optimal conditions.

Indicate the type of investigation : observationnal vs experimental (controled), randomized or not, with the purpose to place the project in the scale of proof.

Make the list of the factors with their range and their essential characteristics (discreet vs continuous, hard-to-change, nested, etc.)

Make the list les responses to be modelled.

Make the list of the models of interest which can be theoretical or empirical.

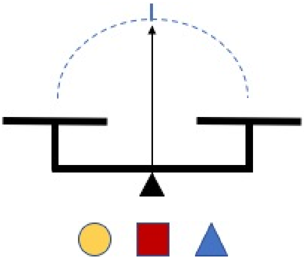
Make the list of the strategies possible to get to the objective. Indicate the number of experiments for each strategy, that will help to take decisions. By linking the elements of the strategy it is possible to define a scenario.

1.4.2 Weighing three objects with a two-pan scale

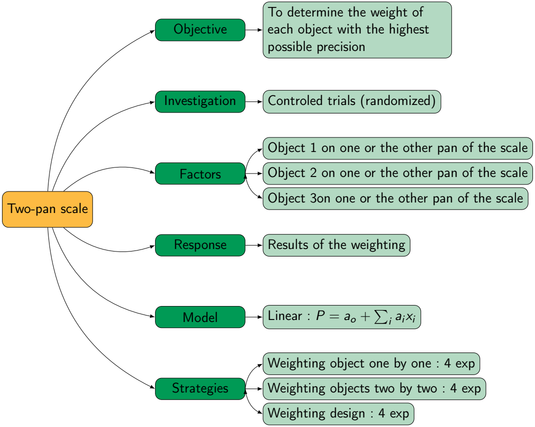
Description of the problem

To measure the weight of three objects with the best accuracy for a reasonable cost :

- The weight of the objects are of the same order of magnitude
- The instrument is a two-pan scale

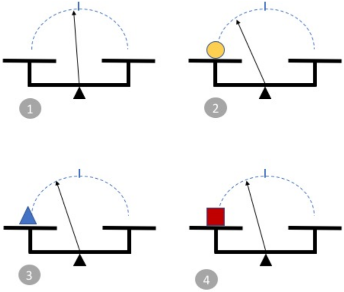


1.4.3 Weighing three objects



1.4.4 Strategy 1 : Weighting the objects one by one

- Four measurements
- One measurement without any object to determine the offset of the scale
- One object only at a time on one of the two pans

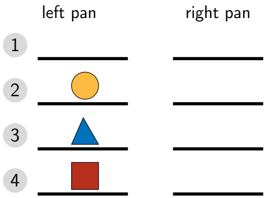


Questions :

- What is the weight of each object ?
- What is the accuracy of the results ?

1.4.4 Strategy 1 : Weighting the objects one by one

- What is the weight of each object ?
- What is the accuracy of the results ?

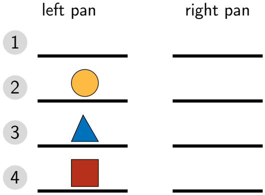


$$\begin{bmatrix} R_o \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} m_o \\ m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

$$\begin{cases} m_o = R_o \\ m_i = R_o - R_i \end{cases}$$

1.4.4 Strategy 1 : Weighting the objects one by one

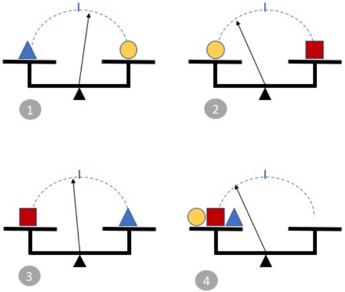
- What is the weight of each object ?
- What is the accuracy of the results ?



$$\begin{cases} var(m_o) & = var(R_o) = \sigma^2 \\ var(m_i) & = var(R_o - R_i) \\ & = var(R_o) + var(R_i) \\ & = 2\sigma^2 \end{cases}$$

1.4.5 Strategy 2 : Weighting objects 2 by 2

- Four measurements
- Three measurements with one object per pan
- One measurement with the three objects on one pan

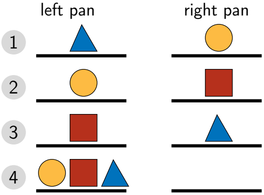


Questions :

- What is the weight of each object ?
- What is the accuracy of the results ?

1.4.5 Strategy 2 : Weighting objects 2 by 2

- What is the weight of each object ?



$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} m_o \\ m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

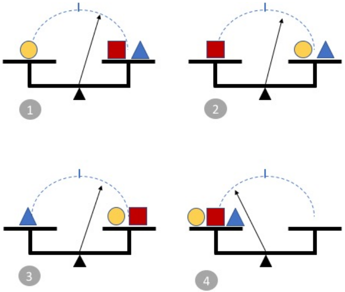
$$\vec{R} = X\vec{m} \quad \Rightarrow \quad \vec{m} = X^{-1}\vec{R}$$

1.4.5 Strategy 2 : Weighting objects 2 by 2

- What is the accuracy of the results ?
- $$\begin{bmatrix} m_o \\ m_1 \\ m_2 \\ m_3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & 3 & 3 & 0 \\ 4 & -2 & 1 & -3 \\ -2 & 1 & 4 & -3 \\ 1 & 4 & -2 & -3 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}$$
- $var(\vec{m}) = var(X^{-1}\vec{R}) = (X^T X)^{-1} var(\vec{R}) = D\ var(\vec{R})$
- $var(m_i) \approx D_{ii}\ \sigma^2$
- $D_{00} = 1/3 \quad D_{11} = D_{22} = D_{33} = 10/27$

1.4.6 Strategy 3 : Weighting objects 3 by 3

- Four measurements
- For three measurements, two objects are weighted against a third one
- For one measurement the three objects are weighted together

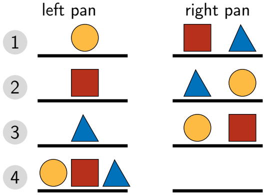


Questions :

- What is the weight of each object ?
- What is the accuracy of the results ?

1.4.6 Strategy 3 : Weighting objects 3 by 3

- What is the weight of each object ?



$$\begin{bmatrix} R_o \\ R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} m_o \\ m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

$$\vec{R} = X\vec{m} \quad \Rightarrow \quad \vec{m} = X^{-1}\vec{R}$$

1.4.6 Strategy 3 : Weighting objects 3 by 3

- What is the accuracy of the results ?
- $$\begin{bmatrix} m_o \\ m_1 \\ m_2 \\ m_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} R_o \\ R_1 \\ R_2 \\ R_3 \end{bmatrix}$$
- $var(\vec{m}) = var(X^{-1}\vec{R}) = (X^T X)^{-1} var(\vec{R}) = D \; var(\vec{R})$
- $var(m_i) \approx D_{ii} \; \sigma^2$
- $D_{00} = D_{11} = D_{22} = D_{33} = 1/4$

1.4.7 Conclusion

- DOE invented in the 20's by Fisher
- Importance of the visual check of data
- Beware of cognitive biases
- Follow the relation between the mathematical and the causal model
- Make a mind-map at the beginning and maintain it along the project
- Sorting data is an easy way of performing visual comparison
- A lot of functions available on Matlab (Python) for data analysis
- The weight of 3 objects : a paradigm of DOE