

9: Relaxation of nuclear magnetization

1. How is the MR signal detected ?
2. What is the quantum-mechanical equivalent of the rotating frame ?
3. What is the rotating frame description good for ?
4. How can the return of the magnetization to thermodynamic equilibrium described ?
5. How is the time-dependent change of magnetization described mathematically ?

After this course you

1. Can describe the principle of MR detection and excitation
2. Can explain how MR excitation is frequency selective (resonance)
3. Understand the principle of relaxation to the equilibrium magnetization
4. Know what are the major relaxation times and how they phenomenologically affect magnetization in biological tissue, in particular that of water.
5. Can explain the elements of the Bloch equations and FID
6. Understand the MR contrast strongly depends on experimental parameters

9-1

Fund Biolmag 2025

What do we know about MR so far ?

Need:

Nucleus with non-zero spin
Magnetic field B_0

Get:

Nuclear (equilibrium) magnetization M_0
(Magnitude dictated by Boltzmann distribution)

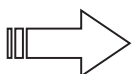
M_0 increases with

1. Number of spins in voxel
2. Magnetic field B_0
3. Gyromagnetic ratio γ

Imaging ^1H in H_2O is most sensitive

Thermodynamic equilibrium
magnetization M_0 is $\parallel B_0$

$$\frac{d\vec{M}_0}{dt} = \vec{M}_0 \times \gamma \vec{B}_0 = 0 \quad M_0 \text{ does not precess}$$



All this does not generate a measurable signal

9-4

Fund Biolmag 2025

9-1. How is the MR signal detected ?

Faraday's Law of Induction

$$\xi = -\frac{d\Phi_B}{dt}$$

Magnetic flux Φ_B

$$\Phi_B = \iint_{\text{surface}} \vec{B}(\vec{r}, t) \cdot d\vec{A}$$



MR: $dA = \text{constant}$

$$\rightarrow \xi \propto dB/dt$$

Lenz's Law

induced voltage $\xi \Rightarrow$ current \rightarrow magnetic field opposes the change in the magnetic flux that produces the current (Completely analogous to power generation!)

Biot-Savart Law

magnetic field falls off with r^2 $\vec{B}(\vec{r}) \propto \int \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3}$

Magnetic field of dipole decreases with distance :
 ξ decreases with distance from magnetization

9-5

9-2. Rotating frame revisited

Equation of motion for \vec{M} (always valid in any reference frame) in presence of B_0

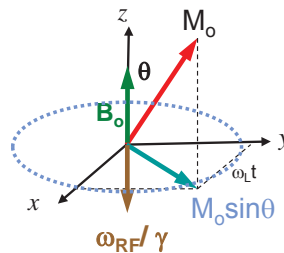
$$\frac{d\vec{M}}{dt} = -\gamma \Delta \vec{B}^{\text{eff}} \times \vec{M}$$

magnetization **precesses** in xy plane with frequency $\gamma \Delta B^{\text{eff}}/2\pi$

Rotating frame: reference frame rotating about z at frequency ω_{RF}

Case I: non-rotating reference frame ($\omega_{\text{RF}}=0$)

\Rightarrow magnetization **precesses** in xy plane with frequency $\gamma B_0/2\pi$



Case II: rotating frame with $\omega_{\text{RF}} = \omega_L$

\Rightarrow magnetization is **stationary** ("precesses" in xy with **zero** frequency)

Equation of motion is still valid, i.e. precession frequency $\gamma \Delta B^{\text{eff}}/2\pi$

$\Rightarrow \Delta B^{\text{eff}} = 0$

Larmor frequency Ω in the rotating frame:

$$\Omega = \gamma \Delta B^{\text{eff}}$$

$$\Delta B^{\text{eff}} = B_0 - \omega_{\text{RF}}/\gamma$$

9-7

Supplement: Rotating frame

What are the quantum-mechanical equivalencies ?

Schrödinger representation:

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = H_S |\psi_S(t)\rangle$$

If $H_S = \text{const}$ in t:

$$|\psi_S(t)\rangle = e^{-iH_S t/\hbar} |\psi_S(0)\rangle$$

NB.

$$\langle I_z \rangle \equiv \langle \psi_S(t) | I_z | \psi_S(t) \rangle$$

Quantum mechanical equivalencies:

$$M_z \propto \langle I_z \rangle, M_x \propto \langle I_x \rangle, M_y \propto \langle I_y \rangle$$

For one spin-1/2 (^1H), i.e. two energy levels

$$I_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad I_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad I_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

How to determine $\langle I_x(t) \rangle$ etc ?

⇒ Split H_S into time-invariant and -dependent terms:

$$i\hbar \frac{d}{dt} |\psi_S(t)\rangle = [H_S^0 + V(t)] |\psi_S(t)\rangle$$

Interaction representation

(Higher order perturbation theory)

$$|\psi_I(t)\rangle \equiv e^{iH_S^0 t/\hbar} |\psi_S(t)\rangle$$



$$i\hbar \frac{d}{dt} |\psi_I(t)\rangle = V_I(t) |\psi_I(t)\rangle$$



$$V_I(t) = e^{iH_S^0 t/\hbar} V_S(t) e^{-iH_S^0 t/\hbar}$$

For spin:

$$H_S^0 = \hbar \gamma B_0 I_z$$

$$V(t) = \hbar \gamma B_1 (\cos(\omega_{RF} t) I_x + \sin(\omega_{RF} t) I_y)$$

What is $V_I(t)$ [$\omega_{RF} = \gamma B_0$]?

$$V_I(t) = \hbar \gamma B_1 I_x$$

Quantum mechanical equivalencies:

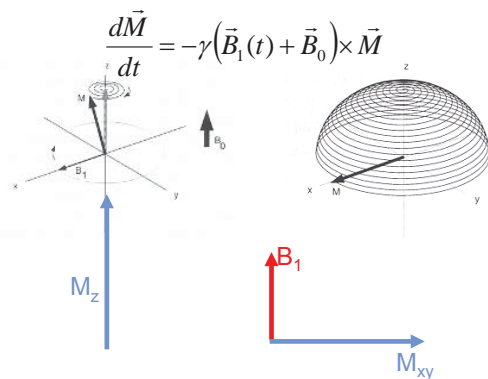
$$B_0 \propto I_z, B_{1x,y} \propto I_{x,y}$$

9-8

Fund Biolmag 2025

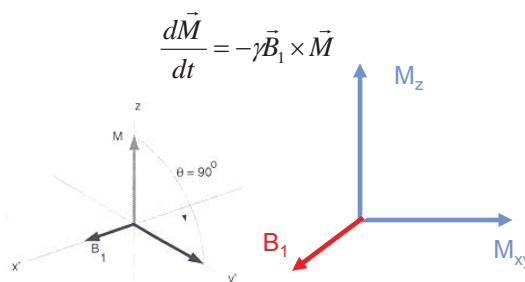
9-3. What is the motion of magnetization when an RF field induces a flip angle ?

Laboratory frame of reference



B_1 radiofrequency field at Larmor frequency ω_L applied in transverse (xy) plane for duration τ
 ⇒ **nutaton** (at ω_L) of M as it tips away from the z-axis.

Rotating frame of reference



RF field rotates M towards xy plane

Amplitude B_1 determines how quickly the magnetization is rotated.

$$\text{flip angle } \alpha = \gamma B_1 \tau \text{ [rad]} \quad \begin{aligned} M_z &= M_0 \cos \alpha \\ M_{xy} &= M_0 \sin \alpha \end{aligned}$$

In MRI typically $\gamma B_1 / 2\pi \sim 0.1\text{-}1\text{kHz}$
 ($\tau \sim 1\text{ms}$)

9-9

Fund Biolmag 2025

What is “resonance” ?

What range of frequencies can be excited with a given RF pulse?

At $\Delta\omega = \omega_L - \omega_{RF}$ (from ω_L) magnetization experiences effective field strength B^{eff}

$$\gamma B^{eff} = \sqrt{(\gamma B_1)^2 + (\Delta\omega)^2}$$

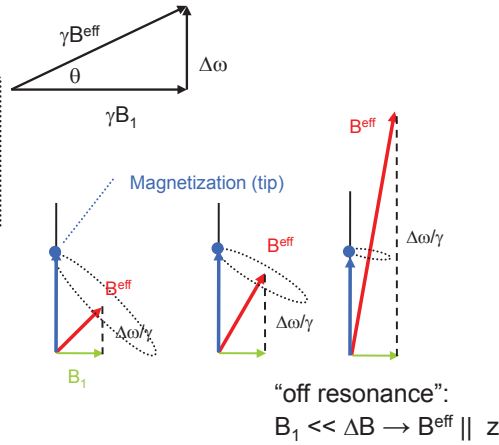
Rotation axis : tilted by θ .

“on resonance”:

$\gamma B_1 \gg \Delta\omega \rightarrow$ effective field $\parallel B_1$

\Rightarrow short RF pulses ($\tau < 1\text{ms}$)

RF field with amplitude B_1 can excite a range of frequencies on the order of $\pm\gamma B_1$



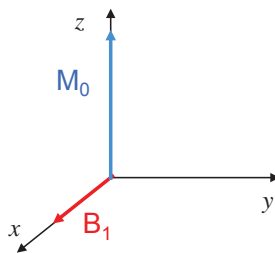
Quantum mechanical “resonance”

Transition probability highest : $h\nu = h\gamma B_0/2\pi$

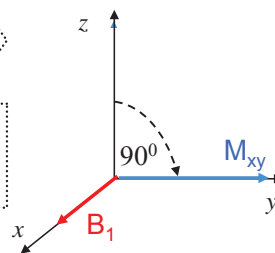
9-4. How is the return to equilibrium M_0 governed ?

Relaxation

Thermodynamic equilibrium



After excitation



Transverse magnetization:

(along x and y-axis, on resonance)

$$\frac{dM_x(t)}{dt} = -\frac{M_x(t)}{T_2}$$

$$\frac{dM_y(t)}{dt} = -\frac{M_y(t)}{T_2}$$

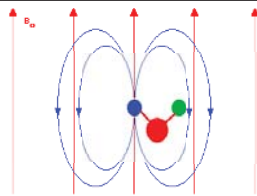
Equations formally equivalent to linear attenuation coefficient (x-ray) (same solution)

Exponential decay of M_{xy}

$$M_{xy}(t) = M_{xy}(0)e^{-\frac{t}{T_2}}$$

What are the mechanisms of relaxation ?

Tumbling of Molecule
(Brownian motion)
Creates local oscillating/fluctuating
magnetic field



Fluctuating magnetic field

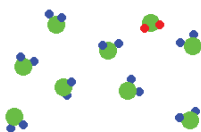


depends on orientation of the whole
molecule & correlation time τ_c (=time for
molecule to rotate 1 rad)

Sources of fluctuating magnetic field:

Dipolar coupling between nuclei and
solvent

interaction between nuclear magnetic
dipoles



Correlation function G

$$G(t) \propto e^{-t/\tau_c}$$

Describes degree of correlation of motion
t sec apart

Correlation time τ_c :

$$\tau_c = \frac{4\pi\eta r^3}{3kT}$$

η : viscosity

k: Boltzmann constant

r: size of molecule

9-12

Fund Biolmag 2025

What is the cause of loss of transverse Magnetization ?

fluctuating *microscopic* magnetic fields δB

T_2 : phenomenological time constant

Range **10 μ s** (bone)... several s (water)

"transverse relaxation", "T2 relaxation"

Cause:

Molecular dynamics and spin-spin
interactions

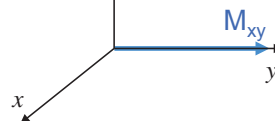
Historically : "spin-spin" relaxation

→ loss of signal in xy plane

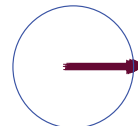
"Memory" relaxation

$$M_{xy}(t) = M_{xy}(0)e^{-\frac{t}{T_2}}$$

After excitation



random precession of nuclei
→ dephasing of spins with
time constant ' T_2 '



Rule of thumb for tissue water:

The less "tissue" (bone, solutes,
proteins, membranes) is in contact with
bulk water, the longer bulk water T_2

Phase ϕ accrued over τ_c :

$$\phi = \delta B \tau_c \quad \iiint_{\text{voxel}} \rho(\vec{r}) e^{i\delta B(\vec{r}, t) \tau_c} dV \rightarrow 0$$

τ_c large (immobile spins):

large phase differences \Rightarrow short T_2

bone, membranes, proteins are MR-"invisible" 9-13

Fund Biolmag 2025

How does M_z return to equilibrium ?

Longitudinal relaxation T_1

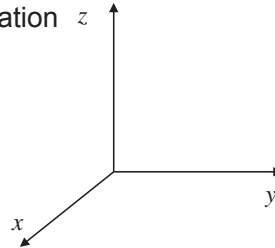
After decay of M_{xy} by T_2 : $M_z \rightarrow M_0$

Longitudinal Relaxation

(along z-axis) $\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_0}{T_1}$

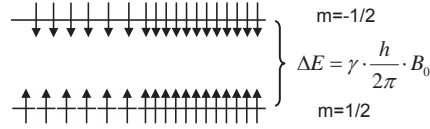
$$M_z(t) = M_0(1 - e^{-t/T_1}) + M_z(0)e^{-t/T_1}$$

After T_2 relaxation z



After 90° excitation: $M_z=0$

\Rightarrow population distribution corresponds to $T=\infty$:



Mechanisms: Incoherent molecular fluctuations on the order of the Larmor frequency ω_L
possibility of energy transfer \rightarrow matching frequency

Historically : spin-lattice relaxation
(heat lost to the surroundings)

$T_1 \sim 0.5-5s$ (water)

$$\frac{N_1}{N_2} = e^{\frac{\Delta E}{kT}}$$

Boltzmann distribution re-established by energy (photon) transfer from spins to system (lattice).

Most efficient when energy levels of system and nuclear spins match, i.e.

$\omega\tau_c \sim 1 \Rightarrow T_1$ minimal

(bone: $\omega\tau_c \gg 1 \Rightarrow T_1 \sim s$ to min)

9-14

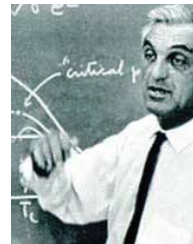
9-5. What equations describe the change in magnetization ?

Bloch Equations

add relaxation terms (T_1 , T_2) to the fundamental Eq of motion of magnetization:

$$\begin{aligned} \frac{dM_z(t)}{dt} &= \gamma[M_x(t)B_y(t) - M_y(t)B_x(t)] - \frac{M_z(t) - M_0}{T_1} \text{ along } z \\ \frac{dM_x(t)}{dt} &= \gamma[M_y(t)B_z(t) - M_z(t)B_y(t)] - \frac{M_x(t)}{T_2} \text{ along } x \\ \frac{dM_y(t)}{dt} &= \gamma[M_z(t)B_x(t) - M_x(t)B_z(t)] - \frac{M_y(t)}{T_2} \text{ along } y \end{aligned}$$

$$- \gamma \vec{B} \times \vec{M}$$



Felix Bloch
Physics
1952

Substituting $\Omega = -\gamma B_0 + \omega_{RF}$ ($B_0 = B_z$ is not time-dependent) yields:

$$\begin{aligned} &\gamma[M_x(t)B_1^y(t) - M_y(t)B_1^x(t)] \\ &- \Omega M_y(t) - \gamma M_z(t)B_1^y(t) \\ &\gamma M_z(t)B_1^x(t) + \Omega M_x(t) \end{aligned}$$

$$(\gamma \vec{B}_1 + \vec{\Omega}) \times \vec{M}$$

B_1 : RF field in rotating frame

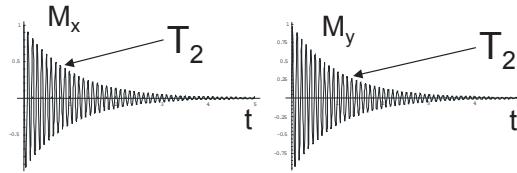
9-15

What characterizes the basic MR signal ?

Free induction decay: Precession and relaxation (after RF pulse)

Transverse magnetization

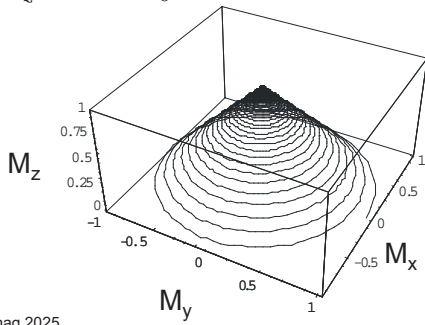
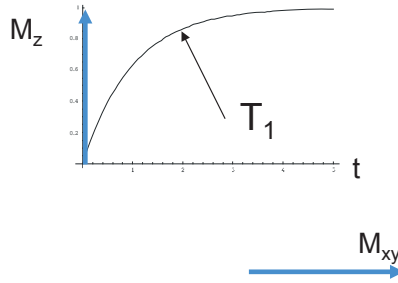
$$M_{xy}(t) = M_{xy}(0)e^{-i\omega t}e^{-t/T_2}$$



Longitudinal magnetization

(after 90° RF excitation)

$$M_z(t) = M_0(1 - e^{-t/T_1})$$



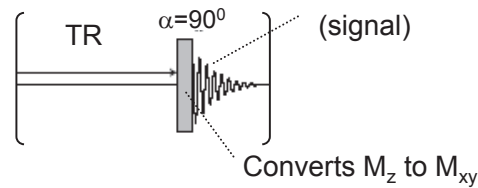
NB. M can never exceed $M_0 \Rightarrow T_2 \leq T_1$

9-16

How can T_1 changes be measured ?

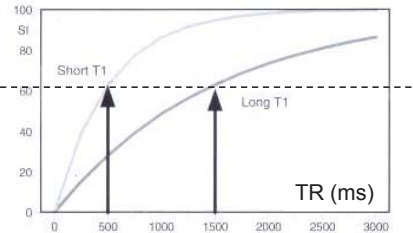
repetitive pulsing

Multipulse experiment with RF pulses applied every TR seconds



$$M_z(TR) = M_0(1 - e^{-TR/T_1})$$

The effect of T_1 (and T_2) on the signal depends on how it is measured



$$M_z(t) = M_0(1 - e^{-t/T_1}) + M(0)e^{-t/T_1}$$

$$M(0) = M_0 \cos \alpha$$

$$\rightarrow M_z(t) = M_0(1 - e^{-t/T_1})$$

Optimal TR to detect changes in T_1 ?

Use noise error propagation calculation (Lesson 1)

$$F = \max: \frac{\partial M_z(t)}{\partial T_1} = \frac{t}{T_1^2} e^{-t/T_1} \equiv F \quad \frac{dF}{dt} = 0 = \frac{1}{T_1^2} e^{-t/T_1} - \frac{t}{T_1^3} e^{-t/T_1}$$

$$0 = \frac{1}{T_1^2} e^{-t/T_1} \left(1 - \frac{t}{T_1} \right) \quad t = TR_{opt} \quad \boxed{TR_{opt} = T_1}$$

9-17

Summary

Magnetic resonance so far

Magnetic field B_0

Equilibrium magnetization $M_0 \parallel z$
Increases with

1. number of spins in voxel
2. Static magnetic field B_0
3. gyromagnetic ratio γ

RF field B_1
(applied on-resonance
i.e. ω_L)

tilts magnetization M into
transverse plane xy

Precession of M_{xy} is detected

T_2 and T_1 relaxation

exponential decay of M_{xy}
exp. return of M_z to M_0

reflect molecular environment
source of contrast

1. Only mobile spins (e.g. water) are detected
2. M_0 reflects amount of nuclei and thus water content
[Water content varies 70-100ml/100g in body (poor contrast)]
3. Effect of T_1 and T_2 changes on image contrast depend strongly
on experimental parameters (RF pulse timing and flip angle)