

Solution 1 - The colour of the sky

Rayleigh scattering through a thicker layer of air causes high energy (blue) photons to be scattered too much to be visible, while lower energy (red) retains more of its directionality and is thus the dominant wavelength (see picture). This effect is also the main reason why the sun appears yellowish to us. From space, it looks actually white.



Rayleigh Scattering:
Picture taken one hour after sunset
(courtesy wikipedia article "Rayleigh Scattering")

Solution 2 - Half-value layer and effective atomic number Z_{eff}

a)

$$n(x) = N_0 e^{-\mu x_a}$$

$$x_a = \frac{-\ln\left(\frac{n(x)}{N_0}\right)}{\mu}; x_a = \frac{-\ln\left(\frac{1}{2}\right)}{(\mu_a \rho N_A)/A} \Rightarrow x_a = 0.94 \text{ cm}$$

b) Use

$$Z_{\text{eff}} = \left(\sum_{i=1}^n \lambda_i Z_i^{3.4} \right)^{\frac{1}{3.4}}, \text{ with } \lambda_i = \frac{P_i Z_i}{A_i}$$

Let <1> refer to H, <2> to O and <3> to Gd:

Z ₁ =1	Z ₂ =8	Z ₃ =64
A ₁ =1	A ₂ =16	A ₃ =157
P ₁ =55·2·1=110	P ₂ =55·16=880	P ₃ =0.001·157=0.157

Plugging this in gives $Z_{\text{eff}}=7.85$, while pure water gives 7.49.

Solution 3 - Compton Scattering

a)

The relationship between energy and wavelength is in this case:

$$hv(\text{keV}) = \frac{hc}{\lambda} = \frac{(6.62 \cdot 10^{-34} \text{ J} \cdot \text{s}) (3 \cdot 10^8 \text{ m/sec})}{(\lambda)(10^{-9} \text{ m/nm}) (1.6 \cdot 10^{-19} \text{ J/eV})(10^3 \text{ eV/keV})} = \frac{1.24}{\lambda}$$

The wavelength λ of a 2-MeV photon then is:

$$\lambda = \frac{1.24}{h\nu} = \frac{1.24}{2000 \text{ keV}} = 0.00062 \text{ nm}$$

The energy transferred to the electron is greatest when the change in wavelength of the photon is maximum; $\Delta\lambda$ is maximum when $\varphi = 180$ degrees.

$$\Delta\lambda_{\text{max}} = 0.00243[1 - \cos(180)] = 0.00243[1 - (-1)] = 0.00486 \text{ nm}$$

The wavelength λ' of the photon scattered at 180 degrees is now:

$$\lambda' = \lambda + \Delta\lambda = (0.00062 + 0.00486) \text{ nm} = 0.00548 \text{ nm}$$

The energy $h\nu'$ of the scattered photon is:

$$h\nu' = \frac{1.24}{\lambda'} = \frac{1.24}{0.00548 \text{ nm}} = 226 \text{ keV}$$

The energy E_k of the Compton electron is:

$$E_k = h\nu - h\nu' = (2000 - 226) \text{ keV} = 1774 \text{ keV}$$

b) $\Delta\lambda = 0.00243 \cdot (1 - \cos\theta)$, so $\Delta\lambda = 0.00071 \text{ nm}$. As above, $h\nu = 1.24/\lambda$, so $\lambda = 1.24/150 = 0.0083 \text{ nm}$. The scattered photon has wavelength $\lambda' = \lambda + \Delta\lambda$, which gives it $\lambda' = 0.00901 \text{ nm}$ and $h\nu' = 138 \text{ keV}$. The Compton electron takes the remaining $150 \text{ keV} - 138 \text{ keV} = 12 \text{ keV}$.

As $\lambda' > \lambda$, the energy of the scattered photon is decreased.

c) λ is the wavelength of the photon **before** scattering,

λ' is the wavelength of the photon **after** scattering,

m is the mass of the electron,

θ is the angle by which the photon's heading changes

Energy and momentum conservation:

$$E_\gamma + E_e = E_{\gamma'} + E_{e'} \quad (1)$$

$$\vec{p}_\gamma = \vec{p}_{\gamma'} + \vec{p}_{e'} \quad (2)$$

where E_γ and p_γ are the energy and momentum of the photon

and E_e and p_e are the energy and momentum of the electron

From (1), we have: $hf + mc^2 = hf' + \sqrt{(p_{e'}c)^2 + (mc^2)^2}$

Solving for $p_{e'}$:

$$(hf + mc^2 - hf')^2 = (p_{e'}c)^2 + (mc^2)^2$$

$$\frac{(hf + mc^2 - hf')^2 - mc^2}{c^2} = p_{e'}^2 \quad (3)$$

Solving (2) and rearrange:

$$\vec{p}_{e'}^2 = \vec{p}_\gamma^2 + \vec{p}_{\gamma'}^2 - 2\vec{p}_{\gamma'}\vec{p}_\gamma = \vec{p}_\gamma^2 + \vec{p}_{\gamma'}^2 - 2|p_{\gamma'}||p_\gamma|\cos\theta$$

$$p_{e'}^2 = \left(\frac{hf}{c}\right)^2 + \left(\frac{hf'}{c}\right)^2 - 2\left(\frac{hf}{c}\right)\left(\frac{hf'}{c}\right)\cos\theta \quad (4)$$

By equating (3) and (4), we get after simplification:

$$-2h^2ff'\cos\theta = -2h^2ff' + 2h(f-f')mc^2$$

Dividing by $-2hff'mc^2$:

$$\frac{f-f'}{ff'} = \frac{h}{mc^2}(1 - \cos\theta)$$

which can be rewritten:

$$\frac{1}{f'} - \frac{1}{f} = \frac{h}{mc^2}(1 - \cos\theta).$$

This is equivalent to the Compton scattering equation, but it is usually written using λ 's rather than f 's.

$$f = \frac{c}{\lambda}, \text{ so we have finally: } \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta).$$

c bis) We can also derive this equation directly from the Compton Energy equation seen in the lecture (slide 3-8):

$$E_f = \frac{E_i}{(1 - \cos\theta) \frac{E_i}{m_e c^2} + 1}$$

For a photon, we have :

$$E = hf = h \frac{c}{\lambda}$$

The Compton equation becomes then:

$$\begin{aligned} (1 - \cos\theta) \frac{E_i}{m_e c^2} E_f + E_f &= E_i \\ \frac{h^2 c^2}{\lambda_i \lambda_f} (1 - \cos\theta) + h \frac{c}{\lambda_f} &= h \frac{c}{\lambda_i} \\ \frac{hc}{\lambda_i \lambda_f m_e c^2} (1 - \cos\theta) + \frac{1}{\lambda_f} &= \frac{1}{\lambda_i} \\ \frac{h}{m_e c} (1 - \cos\theta) + \lambda_i &= \lambda_f \\ \lambda_f - \lambda_i &= \frac{h}{m_e c} (1 - \cos\theta) \end{aligned}$$

Solution 4 - Pair Production

- Subtract 511 keV per created electron and divide the remaining energy by two:
 $(2750-1022)/2=864$ keV.
- $\Delta\lambda = 0.00243 \cdot (1 - \cos\theta)$; if $\phi=60^\circ$, $\Delta\lambda=0.001215$ nm. If we assume that the incident photon has the limit of infinite energy and thus $\lambda=0$, then $\lambda' = \Delta\lambda$ and $h\nu' = 1.24/\lambda' = 1021$ keV. At least 1022 keV is needed for pair production to occur.

Solution 5 - Radiation Protection

- $1\text{Gy} = E/m \Rightarrow 10\text{cGy} = 0.1\text{Gy} = E/(10 \cdot 10^{-3}) \text{ kg} \Rightarrow E = 10^{-3}\text{J}$

- This question is often asked to new people at PET centres.

- i. $\mu = (\mu/\rho) \cdot \rho = 0.1542 \text{ cm}^2/\text{g} \cdot 11.35 \text{ g/cm}^3 = 1.75 \text{ cm}^{-1}$.
- ii. The transmission for perpendicular radiation is 0.25, so $e^{-\mu \cdot d_{25\%}} = e^{-1.75 \text{ cm}^{-1} \cdot d_{25\%}} = 0.25$. This means $\mu \cdot d_{25\%} = \ln(4)$ and thus $d_{25\%} = \ln(4)/1.75 \text{ cm}^{-1} = 0.792 \text{ cm}$.
- iii. $m = V \cdot \rho = A \cdot d_{25\%} \cdot \rho = 1.5 \cdot 10^4 \text{ cm}^2 \cdot 0.792 \text{ cm} \cdot 11.35 \text{ g/cm}^3 = 134838 \text{ g} = 135 \text{ kg}$.
- iv. The calculated weight is too heavy to carry. If we take 10 kg as the maximum apron weight, the transmission would be 92%. This means no lead aprons are used in PET.