**Statistical Physics IV: Non-equilibrium statistical physics**

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

*Solutions to Exercise No.2*

## Solution: Stochastic differential equations

(a) Without the noise process:

$$\begin{aligned}
X\left(t+\frac{dt}{2}+\frac{dt}{2}\right) =& X\left(t+\frac{dt}{2}\right) + A\left(X(t+dt/2),t+dt/2\right)\frac{dt}{2} + \sqrt{D\left(X(t+dt/2),t+dt/2\right)}\sqrt{\frac{dt}{2}} \\
=& X(t) + A(X,t)\frac{dt}{2} + \sqrt{D(X,t)}\sqrt{\frac{dt}{2}} \\
&+ A\left(X(t+dt/2),t+dt/2\right)\frac{dt}{2} + \sqrt{D\left(X(t+dt/2),t+dt/2\right)}\sqrt{\frac{dt}{2}} \\
\approx& X(t) + A(X,t)dt + \sqrt{D(X,t)}\sqrt{2\,dt} \\
\neq& X(t) + A(X,t)dt + \sqrt{D(X,t)}\sqrt{dt}
\end{aligned}$$

(b) With the normal variate $N(0,1)$, which satisfies the property $N(0,1)+N(0,1) = N(0,2) = \sqrt{2}N(0,1)$, the additional numerical factor is cancelled and the Ito equation is self-consistent.

(c) For the mean and 2nd moment, we have the differential equations,

$$\begin{aligned}
\frac{d}{dt}\langle X(t)\rangle &= \langle A(X,t)\rangle \\
\frac{d}{dt}\langle X(t)^2\rangle &= 2\langle A(X,t)X(t)\rangle + \langle D(X,t)\rangle.
\end{aligned}$$

(d) Rewriting the Ito equation as,

$$\begin{aligned}
\frac{X(t+dt)-X(t)}{dt} &= A(X,t) + \sqrt{D(X,t)}\,\frac{N(0,1)}{\sqrt{dt}} \\
&= A(X,t) + \sqrt{D(X,t)}\,N(0,1/dt),
\end{aligned}$$

gives the Langevin equation,

$$\frac{dX}{dt} = A(X,t) + \sqrt{D(X,t)}\,\Gamma(t),$$

where $\Gamma(t) = \lim_{dt\to 0} N(0,1/dt)$; and it thus follows that, $\langle\Gamma(t)\rangle = 0$ and $\langle\Gamma(t)\Gamma(t')\rangle = \delta(t-t')$.

### Solution: Wiener-Khinchin theorem

From quantum noise review appendix 2

$$\begin{aligned}
S_{XX}(\omega) &= \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' e^{i\omega(t'-t)} \langle X(t)X(t')\rangle \\
&= \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-B(t)}^{B(t)} d\tau e^{i\omega\tau} \langle X(t)X(t+\tau)\rangle \quad \text{(change of variable } \tau = t'-t\text{)} \\
&\approx \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle X(t)X(t+\tau)\rangle \quad \text{(ok in limit } T \gg \text{correlation time)} \\
&= \frac{1}{T}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle X(0)X(0+\tau)\rangle \quad \text{(by time invariance)} \\
&= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \langle X(0)X(\tau)\rangle = \int_{-\infty}^{\infty} e^{i\omega\tau} C_{XX}(\tau) d\tau
\end{aligned}$$

### Solution: Review of Spectral Densities

(a) For an O-U process, the random variable $X(t)$ is classical and takes real values. Moreover, the process is stationary so we can write

$$\begin{aligned}
C_{XX}(-\tau) &= \langle X(t)X(t-\tau)\rangle \\
&= \langle X(t'+\tau)X(t')\rangle \\
&= \langle X(t')X(t'+\tau)\rangle \\
&= \langle X(t)X(t+\tau)\rangle = C_{XX}(\tau)
\end{aligned}$$

(b)

$$\begin{aligned}
S_{XX}(\omega) &= \frac{1}{2\pi}\int C_{XX}(\tau)e^{i\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int C_{XX}(-\tau)e^{i\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int C_{XX}(\tau')e^{i\omega(-\tau')}d\tau' = S_{XX}(-\omega)
\end{aligned}$$

This also implies the function is real since $S_{XX}(\omega)^* = S_{XX}(-\omega)$ for real $C_{XX}$.

The non-negativity is most easily seen from the Wiener-Khinchin theorem.

(c)

$$\begin{aligned}
S_{XX}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\tau)(\cos(\omega\tau) + i\sin(\omega\tau))d\tau \\
&= \frac{1}{2\pi}2\int_{0}^{\infty} C(\tau)\cos(\omega\tau)d\tau \\
&= \frac{1}{2}\left[4\frac{1}{2\pi}\int_{0}^{\infty} C(\tau)cos(\omega\tau)d\tau\right] = \frac{1}{2}S_{XX}^{\text{single}}(\omega)
\end{aligned}$$

(d) For an O-U process we have $\langle X(t)X(t+t')\rangle = \langle X^2(t)\rangle e^{-t'/\tau}$ so

$$\begin{aligned}
S_{XX}(\omega) &= \frac{1}{2\pi}\int \langle X^2(t)\rangle e^{-t'/\tau}e^{-i\omega t'}dt' \\
&= \frac{\langle X^2(0)\rangle}{2\pi}\frac{2\tau}{1+(\omega\tau)^2} \\
&= \frac{c\tau/2}{2\pi}\frac{2\tau}{1+(\omega\tau)^2}
\end{aligned}$$

and $S_{XX}^{\text{single}}(\omega) = \frac{\langle X^2(0) \rangle}{2\pi} \frac{4\tau}{1 + i\omega\tau}$

**Solution: Johnson noise and its spectrum**

(a) The equation is of the form of the Langevin equation, with $\tau = L/R$ and $V(t) = L\sqrt{c}\Gamma(t)$. For the limit of large time, the equipartition applies, so that

$$\left\langle \frac{1}{2} L I^2 \right\rangle = \frac{1}{2} k_B T$$

and the Ornstein-Uhlenbeck variance tends to

$$\langle I^2(t \to \infty) \rangle = \frac{c\tau}{2}$$

so we get $c = 2k_B TR/L^2$

(b) $V(t) = \sqrt{2kTR}\Gamma(t)$, so we get our result.

(c) $S_{VV}(\omega)$ is the Fourier transform of $2kTR\delta(\tau)$, so we get the result.