Stat. Phys. IV: Lecture 9

Spring 2025

Problem formulation: an example

Consider the (classical) ideal harmonic oscillator with the following equation of motion (q denotes position):

$$\ddot{q} + \omega^2 q = 0 \; .$$

To add damping to the system we simply add a term $-\gamma\dot{q}$ on the right-hand side:

$$\ddot{q} + \gamma\dot{q} + \omega^2 q = 0 \; .$$

Now consider the quantum mechanical case (with damping):

$$\left[\hat{q}, \hat{p}\right] = i\hbar \; ; \; \dot{\hat{q}} = \frac{\hat{p}}{m} \; ; \; \dot{\hat{p}} = -m\omega^2\hat{q} - \gamma\hat{p} \; .$$

We can calculate the derivative of the commutator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{q}, \hat{p}\right] = -\gamma\left[\hat{q}, \hat{p}\right] \text{ so } \left[\hat{q}, \hat{p}\right] = i\hbar e^{-\gamma t} \; .$$

In other words the commutator decays with time! This violates the Heisenberg uncertainty principle!

Examples of open quantum systems

In what contexts does damping quantum systems play a role?

1. Laser systems / Optical cavities
2. Lossy LC circuits
3. Spontaneous emission

System plus reservoir approach

Let's take a simple example:

$$\hat{H}_{\mathsf{sys}} + \hat{H}_{\mathsf{bath}} = \underbrace{\hbar\omega\hat{a}^{\dagger}\hat{a}}_{\text{SYSTEM}} + \underbrace{\sum_k \hbar\omega_k \hat{b}_k^{\dagger}\hat{b}_k}_{\text{BATH}} \ .$$

We can introduce an interaction between system and bath of the following form (in the rotating wave approximation):

$$\hat{H}_{\mathsf{int}} = \sum_k \hbar g_k \left(\hat{a}^{\dagger}\hat{b}_k + \hat{a}\hat{b}_k^{\dagger}\right) \ .$$

System plus reservoir approach (2)[1]

Let's solve the equations of motion for the Hamiltonian $\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{int}} + \hat{H}_{\text{bath}}$ from the previous page.

$$\begin{aligned}
\dot{\hat{a}} &= \frac{i}{\hbar}[H, \hat{a}] = -i\omega\hat{a}(t) - i\sum_k g_k \hat{b}_k(t) \\
\dot{\hat{b}}_k &= i\omega_k \hat{b}_k - i g_k \hat{a}(t)
\end{aligned}$$

We now formally integrate the equation for $\hat{b}_k$ and insert back into the equation for $\hat{a}$.

$$\begin{aligned}
\hat{b}_k(t) &= \hat{b}_k(0)e^{-i\omega_k t} - i g_k \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{-i\omega_k(t-t')} \\
\dot{\hat{a}}(t) &= -i\omega\hat{a} - \sum_k g_k^2 \int_0^t \mathrm{d}t'\, \hat{a}(t')e^{-i\omega_k(t-t')} + \hat{f}_a(t) \\
\hat{f}_a(t) &= -i\sum_k g_k \hat{b}_k(0)e^{-i\omega_k t}
\end{aligned}$$

[1] See Scully "Quantum Optics"

Quantum Langevin Equation

First we move to a rotating reference frame: $\hat{\tilde{a}} = \hat{a}(t)e^{i\omega t}$.
Then we simplify the expressions by using the Markov approximation (assuming $g(\omega)$ to be frequency independent) and assuming the mode spacing is small (see next slide):

$$\sum_k g_k^2 \int_0^t \mathrm{d}t' \, \hat{\tilde{a}}(t') e^{-i(\omega_k - \omega)(t-t')} \approx \frac{1}{2}\kappa \hat{\tilde{a}}(t) \ , \ \kappa = 2\pi \left| g(\omega) \right|^2 D(\omega) \ ,$$

where $D(\omega)$ is the density of states. In the original frame we obtain

QUANTUM LANGEVIN EQUATION

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -i\omega\hat{a}(t) - \frac{\kappa}{2}\hat{a}(t) + \hat{f}_a(t)$$

Mathematical details

In the previous slide we first converted the summation over k to an integral:

$$\sum_k g_k^2 = \int_0^\infty \mathrm{d}\omega_k \, D(\omega_k) \left| g(\omega_k) \right|^2 ,$$

where $D(\omega)$ is the density of states. Secondly we assumed that the coupling g_k is approximately frequency independent. This is called the 1st Markov approximation.

$$\begin{aligned}
\sum_k g_k^2 \int_0^t \mathrm{d}t' \, \hat{\tilde{a}}(t') e^{-i(\omega_k-\omega)(t-t')} &= \int_0^\infty \mathrm{d}\omega_k \, D(\omega_k) \left| g(\omega_k) \right|^2 \int_0^t \mathrm{d}t' \, \hat{\tilde{a}}(t') e^{-i(\omega_k-\omega)(t-t')} \\
&\cong D(\omega) \left| g(\omega) \right|^2 \int_0^t \hat{\tilde{a}}(t') \int_0^\infty \mathrm{d}t' \, \mathrm{d}\omega_k \, e^{-i(\omega_k-\omega)(t-t')} \\
&\cong D(\omega) \left| g(\omega) \right|^2 \int_0^t \mathrm{d}t' \, \hat{\tilde{a}}(t') 2\pi\delta(t-t') \\
&= 2\pi D(\omega) \left| g(\omega) \right|^2 \frac{1}{2} \hat{\tilde{a}}(t) \equiv \frac{\kappa}{2} \hat{\tilde{a}}(t)
\end{aligned}$$

Quantum to Classical, Assumptions about the bath modes

For a thermal state described by $\rho = \rho_{nn}$ and $\rho_{nn} = \frac{e^{-\beta E_n}}{e^{-\beta H}}$ (the off-diagonal elements vanish in a thermal state):

$$\left\langle \hat{b}_k(0) \right\rangle_R = \left\langle \hat{b}_k^\dagger(0) \right\rangle_R = 0 \,, \qquad \left\langle \hat{b}_k^\dagger(0)\hat{b}_{k'}(0) \right\rangle_R = \bar{n}_k \delta_{k,k'} \,,$$
$$\left\langle \hat{b}_k(0)\hat{b}_{k'}^\dagger(0) \right\rangle_R = (\bar{n}_k + 1)\delta_{k,k'} \,, \quad \left\langle \hat{b}_k^\dagger(0)\hat{b}_k^\dagger(0) \right\rangle_R = \left\langle \hat{b}_k(0)\hat{b}_k(0) \right\rangle_R = 0 \,.$$

Index R means averaging over the reservoir modes. Let's calculate $\left\langle \hat{F}(t) \right\rangle = 0$ and:

$$\begin{aligned} \left\langle \hat{F}^\dagger(t)\hat{F}(t') \right\rangle &= \sum_k \sum_{k'} g_k g_{k'} \left\langle \hat{b}_k^\dagger \hat{b}_{k'} \right\rangle e^{-i(\omega_k - \omega)t + i(\omega_{k'} - \omega)t'} \\ &= \int_0^\infty D(\omega_k) \left| g(\omega_k) \right|^2 \mathrm{d}\omega_k \, e^{i(\omega_k - \omega)(t - t')} \bar{n}(\omega_k) \\ &= \kappa \bar{n}(\omega) \delta(t - t') \,. \end{aligned}$$

We notice that this is in perfect analogy with the classical Langevin equations:

$$\left\langle \hat{F}(t)\hat{F}^\dagger(t) \right\rangle = \kappa[\bar{n}(\omega) + 1]\delta(t - t') \,, \; \left\langle \hat{F}(t)\hat{F}(t') \right\rangle = 0 \,.$$

Quantum theory of damping

$$\partial_t \left\langle \hat{\tilde{a}}^\dagger(t)\hat{\tilde{a}}(t) \right\rangle = \partial_t \left\langle n(t) \right\rangle = \left\langle \partial_t(\hat{\tilde{a}}^\dagger(t))\hat{\tilde{a}}(t) \right\rangle + \left\langle \hat{\tilde{a}}^\dagger(t)\partial_t(\hat{\tilde{a}}(t)) \right\rangle$$
$$\partial_t \left\langle \hat{\tilde{a}}(t)\hat{\tilde{a}}^\dagger(t) \right\rangle = -\kappa \left\langle \hat{\tilde{a}}(t)\hat{\tilde{a}}^\dagger(t) \right\rangle + \kappa(\bar{n}(\omega) + 1)$$

We see that the energy of the system decays twice as fast as the amplitude of our operators. If we check the commutator again:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\tilde{a}}(t)\hat{\tilde{a}}^\dagger(t)) - \frac{\mathrm{d}}{\mathrm{d}t}(\hat{\tilde{a}}^\dagger(t)\hat{\tilde{a}}(t)) \\ &= -\kappa\left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] + \kappa = 0 \; . \end{aligned}$$

Thus, the commutator is preserved:

$$\left[\hat{\tilde{a}}(t), \hat{\tilde{a}}^\dagger(t)\right] = 1 \; .$$

Quantum theory of damping(2)

$$\dot{\hat{a}}(t) = -\frac{\kappa}{2}\hat{a}(t) + \hat{F}(t)$$

Fourier transform (spectrum):

$$S_{aa}(\omega) = \frac{\langle n \rangle}{\pi} \frac{\kappa/2}{(\omega - \omega_c)^2 + \kappa^2/4}$$

Quantum Langevin Equations for a two level system (1)

Pauli matrices:

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let's assume our two level system is coupled to an electromagnetic field with infinitely many modes:

$$\hat{H}_{\text{sys}} = \frac{1}{2}\hbar\Omega\hat{\sigma}_z \ , \ \hat{H}_{\text{bath}} = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \ ,$$

$$\hat{H}_{\text{int}} = \sum_k \hbar g_k \left(\hat{\sigma}_- \hat{b}_k^\dagger + \hat{\sigma}_+ \hat{b}_k \right)$$

Using the same procedure as previously (c.f. homework) we obtain:

$$\begin{aligned}
\partial_t \hat{\sigma}_- &= -i\Omega\hat{\sigma}_- - \frac{\kappa}{2}\hat{\sigma}_- + \sqrt{\kappa}\hat{\sigma}_z \hat{b}_{\text{in}}(t) \\
\partial_t \hat{\sigma}_+ &= i\Omega\hat{\sigma}_+ - \frac{\kappa}{2}\hat{\sigma}_+ + \sqrt{\kappa}\hat{\sigma}_z \hat{b}_{\text{in}}(t) \\
\partial_t \hat{\sigma}_z &= -\kappa(1 + \hat{\sigma}_z) - 2\sqrt{\kappa}\left(\hat{\sigma}_+ \hat{b}_{\text{in}}(t) + \hat{b}_{\text{in}}^\dagger(t)\hat{\sigma}_- \right)
\end{aligned}$$

Quantum Langevin Equations for a two level system (2)

The above equations are not closed and simplify significantly if we assume that the atom is initially in the excited state $|2\rangle$ and that the field is in the vacuum state $\left\langle \hat{b}_k^\dagger(t)\hat{b}_k(t)\right\rangle = 0$ at time $t = t_0$. Then

$$\begin{aligned}
\langle \partial_t \hat{\sigma}_- \rangle &= \left(-i\Omega - \frac{\kappa}{2}\right)\langle \hat{\sigma}_- \rangle \ , \\
\langle \partial_t \hat{\sigma}_+ \rangle &= \left(i\Omega - \frac{\kappa}{2}\right)\langle \hat{\sigma}_+ \rangle \ , \\
\langle \partial_t \hat{\sigma}_z \rangle &= -\kappa\left(1 + \langle \hat{\sigma}_z \rangle\right) \ ,
\end{aligned}$$

where we have assumed $\left\langle \hat{\sigma}_z \hat{b}_{\text{in}}(t)\right\rangle = \langle \hat{\sigma}_z \rangle \left\langle \hat{b}_{\text{in}}(t)\right\rangle$ (i.e. the system is in a separable state). Hence the atom decays as:

$$\left\langle \hat{\sigma}_z(t)\right\rangle = \hat{\sigma}_z(0)e^{-\kappa t}, \kappa = 2\pi\left|g_k\right|^2 D(\omega) \ .$$

Here $D(\omega) = \frac{\omega^2}{c^3\pi^2}$ is the density of states of the electromagnetic field. We obtain a rigorous expression for the atomic decay $\kappa = \frac{\omega^3 e^2 \langle 1|\vec{r}|2\rangle}{3\pi\epsilon_0 c^3}$.

Nobel Prize 2012 - Serge Haroche

$$
\begin{aligned}
\hat{H}_{\text{cav}} &= \hbar\omega_c \hat{a}^\dagger \hat{a} \\
\hat{H}_{\text{atom}} &= \frac{1}{2}\hbar\Omega\hat{\sigma}_z \\
\hat{H}_{\text{bath}} &= \hbar\sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k \\
\hat{H}_{\text{int}}^{\text{atom}} &= \hbar g(\sigma^+\hat{a} + \sigma^-\hat{a}^\dagger) \\
\hat{H}_{\text{int}}^{\text{cav}} &= \hbar\sum_k g_k^c(\hat{a}\hat{b}_k^\dagger + \hat{a}^\dagger\hat{b}_k)
\end{aligned}
$$

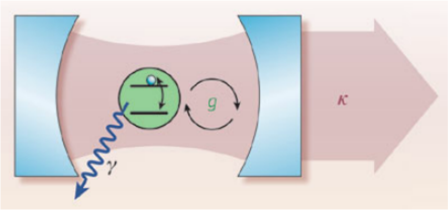

Figure: R.J. Schoelkopf, S.M. Girvin, Nature **451**, 664-669 (2008)

We can consider a simple case where $\hbar\omega \gg k_B T$ (i.e. $\bar{n}_{\text{th}} \simeq 0$). Then, under the weak coupling condition $g \ll \kappa$:

$$\langle \hat{\sigma}_z(t) \rangle = -1 + 2e^{-4g^2 t/\kappa} .$$

The atomic decay rate is thus $\Gamma_c = \frac{4g^2}{\kappa}$. This enhancement of the spontaneous emission rate (i.e. it is faster than in free space) is called the Purcell Effect (Phys. Rev. **69**, 37 (1946)).