Stat. Phys. IV: lecture 6

Spring 2025

1D random walk

- Probability distribution for each step $y_n = x_{n+1} - x_n$ follows:

$$P(y) = \frac{1}{\sqrt{4\pi D t}} \exp(-\frac{y^2}{4Dt}) \tag{1}$$

- After n steps at time $t = n\tau$ (τ is the time increament), the position $x_n = \sum_{i=1}^{n} y_i$ has the probability distribution:

$$P(x, n) = \frac{1}{\sqrt{4\pi D\tau n}} \exp(-\frac{x^2}{4D\tau n}) \tag{2}$$

- It's not stable since $\langle x(n)^2 \rangle = 2D\tau n \to \infty$ as $n \to \infty$
- However the probability distribution of the scaled variable $u_n = \frac{x_n}{\sqrt{n}}$ is not dependent on n:

$$P(u, n) = \sqrt{n} \cdot \frac{1}{\sqrt{4\pi D\tau n}} \exp(-\frac{u_n^2}{4D\tau}) = \frac{1}{\sqrt{4\pi D\tau}} \exp(-\frac{u_n^2}{4D\tau}) \tag{3}$$

Lévy-flights, the Cauchy-distribution

- Lévy-flight is a random walk for which the length of the steps follow a heavy-tailed probability distribution.
- More precisely, consider a positive-valued homogeneous Markov process x_n and define the increments as $y_n = x_{n+1} - x_n$. Assume

$$P(y) \approx \frac{b}{y^{1+\mu}} \quad \text{for y} \to \infty,\ \mu > 0 \tag{4}$$

- For $0 < \mu \leq 2$, the second moment $\langle y^2 \rangle$ diverges and the distribution is heavy-tailed (note: for $0 < \mu \leq 1$ the first moment diverges as well)
- $\mu = 1$ corresponds to the Cauchy-distribution, which has the exact form

$$P(y) = \frac{2b}{\pi(y^2 + b^2)} \quad y \geq 0 \tag{5}$$

The central limit theorem

- Question: how are $x_N = \sum_{i=1}^{N} y_n$ (Lévy sums) distributed in the limit of large N?

If $\mu > 2$: central limit theorem (CLT) applies

- In this case both the mean $\langle y \rangle$ and the variance σ exist. Define $\xi = \frac{x_N - \langle y \rangle N}{\sigma \sqrt{N}}$
- CLT states:

$$\lim_{N \to \infty} \mathcal{P}\left(\xi_1 \leq \xi \leq \xi_2\right) = \int_{\xi_1}^{\xi_2} G(\xi) \mathrm{d}\xi$$

where $G(\xi) = (2\pi)^{-1/2} \exp(-\xi^2/2)$ is the "normal" distribution.

- For all the other cases, i.e. when $\mu \leq 2$, the CLT does not apply!

The generalized central limit theorem, 2 cases

- We discuss the two separate cases $1 < \mu < 2$ and $0 < \mu < 1$. The cases $\mu = 1$ or $\mu = 2$ require special discussion (logarithmic correction factors apply).

The case of $1 < \mu < 2$

- In this case the mean $\langle y \rangle$ is finite but the variance σ diverges. Define $\xi = \frac{x_N - \langle y \rangle N}{y_b N^{1/\mu}}$
- Generalized CLT states:

$$\lim_{N \to \infty} \mathcal{P}\left(\xi_1 \leq \xi \leq \xi_2\right) = \int_{\xi_1}^{\xi_2} L_\mu(\xi) \mathrm{d}\xi$$

  where $L_\mu(\xi)$ is the Lévy distribution of index μ.
- $L_\mu(\xi)$ have simple Laplace transforms: $\mathcal{L}L_\mu(\xi) = \exp(-b_\mu u^\mu)$ with $b_\mu = \mu^{-1}(\mu - 1)\Gamma(1 - \mu)$

The generalized central limit theorem, 2 cases

- We discuss the two separate cases $1 < \mu < 2$ and $0 < \mu < 1$. The cases $\mu = 1$ or $\mu = 2$ require special discussion (logarithmic correction factors apply).

The case of $0 < \mu < 1$

- In this case neither the mean $\langle y \rangle$ nor the variance σ is finite. Define $\xi = \frac{x_N}{y_b N^{1/\mu}}$
- Generalized CLT states:

$$\lim_{N\to\infty} \mathcal{P}\left(\xi_1 \leq \xi \leq \xi_2\right) = \int_{\xi_1}^{\xi_2} L_\mu(\xi) \mathrm{d}\xi$$

  where $L_\mu(\xi)$ is the Lévy distribution of index μ.
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How rare events dominate statistics

Probability to observe an increment $y > \bar{y}$ is: $Q(\bar{y}) = \int_{\bar{y}}^{\infty} P(y)\, dy$

Since probability to observe $y < \bar{y}$ for $(n-1)$ times is $(1 - Q(\bar{y}))^{n-1}$, it follows:

$$Q(\bar{y}, n) = n\, Q(\bar{y})\; [1 - Q(\bar{y})]^{n-1}\,.$$

Most probable value $\bar{y}_{\mathsf{max}}$ is determined by $\frac{dQ(\bar{y},n)}{d\bar{y}} = 0$;

$$\begin{aligned} \frac{dQ(\bar{y}, n)}{d\bar{y}} &= n\frac{dQ(\bar{y})}{d\bar{y}}(1 - Q(\bar{y}))^{n-1} - nQ(\bar{y})\frac{dQ(\bar{y})}{d\bar{y}}(1 - Q(\bar{y}))^{n-2}(n-1) \\ &= n\frac{dQ(\bar{y})}{d\bar{y}}(1 - Q(\bar{y}))^{n-2}(1 - nQ(\bar{y})) = 0 \end{aligned}$$

this implies

$$\boxed{Q(\bar{y}_{\mathsf{max}}) = \frac{1}{n}.}$$

How rare events dominate statistics

For Brownian motion, with Gaussian pdf,

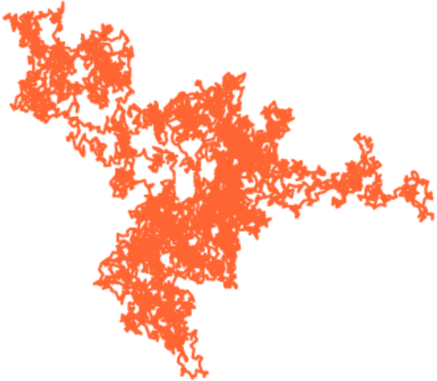
$$\lim_{n\to\infty} \frac{\bar{y}_n}{\sqrt{\langle x_n^2 \rangle}} \approx \frac{\sigma\sqrt{2\ln n}}{\sqrt{n}} \to 0.$$

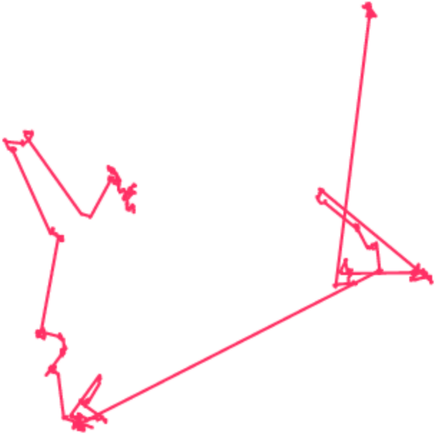
For a Lévy flight, with $P(y) = \frac{b}{y^{1+\mu}}$,

$$\lim_{n\to\infty} \frac{\bar{y}_n}{\sqrt{\langle x_n^2 \rangle}} \approx \frac{n^{1/\mu}}{n^{1/\mu}} \to 1.$$

Thus, rare events dominate the statistics of Lévy flights.

Lévy flight versus Brownian motion[1]





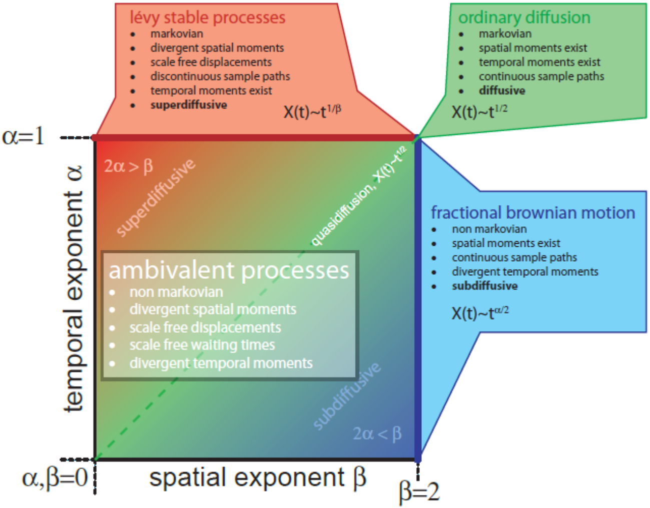
[1] **brockmann_scaling_2006**.

Continuous time random walks

Consider processes for which $P(y, \Delta t) = f(y)\Phi(\Delta t)$ where both $f$and Φ are probability density distributions.

Process	f	Φ	$P(x, \Delta t)$	Scaling
Ordinary diffusion	gaussian	exponential	$\frac{e^{-x^2/Dt}}{\sqrt{t}}$	$t^{1/2}$
Levy flight	$\frac{1}{y^{\beta+1}}$	*	$\frac{L_\beta(x/t^{1/\beta})}{t^{1/\beta}}$	$t^{1/\beta}$
Fractional brownian motion	gaussian	$\frac{1}{(\Delta t)^{\alpha+1}}$	$\frac{L_\beta(x/t^{\alpha/2})}{t^{\alpha/2}}$	$t^{\alpha/2}$
Ambivalent process	$\frac{1}{y^{\beta+1}}$	$\frac{1}{(\Delta t)^{\alpha+1}}$		$t^{\alpha/2\beta}$

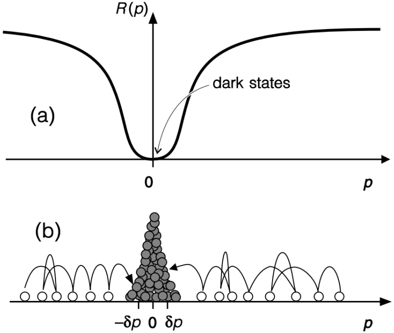
Asymptotic Universality Classes of CTRWs[2]



[2] **brockmann_scaling_2006**.

Application: sub-recoil laser cooling

- During the interaction time with the laser field, an atom will be trapped for some time intervals τ_i and will be in the recycling region for some other intervals $\hat{\tau}_i$
- The sums formed by summing the trapping times, i.e. $\tau_N = \sum_{i=1}^{N} \tau_i$ are Lévy sums
- Main idea: construct a momentum dependent fluorescence rate $R(p)$ in the trapping region such that $R(p) \approx 0$ around $p \approx 0$



Application: sub-recoil laser cooling

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Subrecoil Laser Cooling and Lévy Flights

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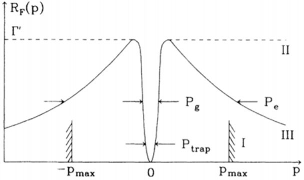


FIG. 2. Variations with p of the fluorescence rate $R_F(p)$ (see text). The narrow dip around $p=0$, with a width p_g, is due to VSCPT. The trapping zone is defined by $|p|<p_{\rm trap}$. Three different models are taken for the variations of $R_F(p)$ at large p. Model I: walls confining the atomic momentum to $|p| \leq p_{\max}$. Model II: constant fluorescence rate equal to Γ' out of the dip (interrupted line). Model III (corresponding to actual experiments, full line): decrease of the fluorescence rate for $|p|>p_e$, due to a Doppler detuning from the optical resonance.

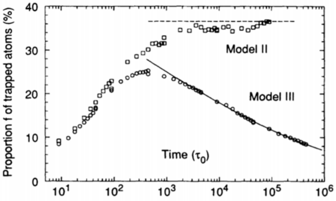
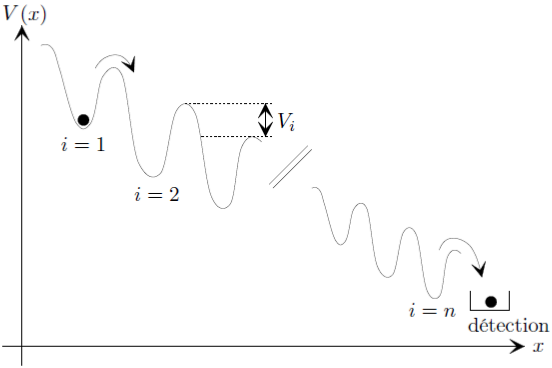


FIG. 3. Variations with the interaction time θ of the proportion f of trapped atoms ($|p|<p_{\rm trap}$), calculated from N Monte Carlo runs for the models II and III of Fig. 2. Model II (squares): $N=4000$, $p_g=0.5\hbar k$, $p_{\rm trap}=0.08\hbar k$; the interrupted line represents the asymptotic theoretical prediction $f=0.365$ corresponding to $E_R/\hbar\Gamma'=0.59$. Model III (circles): $N=16000$, $p_g=0.5\hbar k$, $p_{\rm trap}=0.08\hbar k$, $p_e=9.4\hbar k$; the full line represents the best fit for the asymptotic theoretical prediction (see text). Model II requires more computer time than model III. This is why N is smaller and the statistical uncertainty larger.

Arrhenius Cascade

A good example of the applications of Lévy flights is the Arrhenius cascade (c.f. Martin, 3.8.4).



For exponentially distributed potential barrier heights, the probability distribution of the total time it takes to cross n barriers is distributed according to:

$$P(\tau) = \mu \frac{\tau_0^{\mu}}{\tau^{1+\mu}} \ ,$$

Questions for the paper presentation

- Explain the basic principles of atomic laser cooling.
- Explain what is a ”dark” state and why it is important to the cooling.
- What process is simulated in Fig1, and how is it simulated? Why does the simulation indicate the interval distribution is long-tailed?