Stat. Phys. IV: Lecture 4

Spring 2025

Generalized Fluctuation-Dissipation relation[1]

A process with memory (frequency dependent damping):

$$\dot{v} = -\int_0^t \gamma(t-s)v(s)\,ds + \frac{F(t)}{m}$$

$$\Rightarrow\ v[\omega] = \chi[\omega]\cdot F[\omega]; \qquad \chi[\omega] = [m(i\omega + \gamma[\omega])]^{-1}\ \text{(admittance)}$$

The Generalized FDT states:

$$m\gamma[\omega] = \frac{1}{k_B T}\int_{-\infty}^{\infty} \langle F(t)F(t+\tau)\rangle e^{-i\omega\tau}\,d\tau$$

- Example: radiation reaction force on a charge implies fluctuating electric field satisfying the Planck radiation law

[1] HB Callen and TA Welton. "Irreversibility and generalized noise". In: *Phys. Rev.* 83 (1951), p. 34, R Kubo. "The fluctuation-dissipation theorem". In: *Rep. Prog. Phys.* 29.1 (1966), p. 255.

Proving the GFDT – recapitulation of the steps

1. Formally solve the equation of motion:

$$v(t) = v(0)\cdot m\chi(t) + \int_0^t \chi(t-t')F(t')\,dt'$$

2. Using equipartition ($m\langle v(0)^2\rangle = k_B T$), and $\langle v(0)F(t')\rangle = 0$, gives:

$$\langle v(t)v(0)\rangle = k_B T\,\chi(t)$$

3. Equation of motion and Wiener-Khinchin theorem, gives:

$$\int_{-\infty}^{\infty}\langle v(t)v(0)\rangle e^{-i\omega t}\,dt = |\chi[\omega]|^2\int_{-\infty}^{\infty}\langle F(t)F(0)\rangle e^{-i\omega t}\,dt$$

4. Rearranging using the symmetry of the correlator, then gives:

$$\int_{-\infty}^{\infty}\langle F(t)F(0)\rangle e^{-i\omega t}\,dt = 2k_B T\,\mathrm{Re}\left[\chi[\omega]^{-1}\right]$$

GFDT in the quantum limit[2]

$$\langle V^2 \rangle = \frac{2}{\pi} \int_0^\infty R(\omega) \left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \right) d\omega$$

reducing in the classical limit to

$$\langle V^2 \rangle \approx \frac{2}{\pi} k_B T \int_0^\infty R(\omega) d\omega$$

- V: fluctuating force
- $R(\omega)$: real part of impedance $Z(\omega)$
- $Z(\omega)$: impedance defined as $V = Z(\omega)\dot{Q}$ with response $\dot{Q}$

[2] HB Callen and TA Welton. "Irreversibility and generalized noise". In: *Phys. Rev.* 83 (1951), p. 34.

Application: quantum noise of Josephson junctions[3]

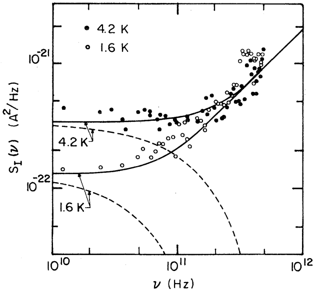


FIG. 6. Measured spectral density of current noise in shunt resistor of junction 2 at 4.2 K (solid circles) and 1.6 K (open circles). Solid lines are prediction of Eq. (1.4), while dashed lines are $(4h\nu/R)[\exp(h\nu/k_BT)-1]^{-1}$.

$$S_I(\nu) = \frac{4h\nu}{R}\left(\frac{1}{\exp(h\nu/k_BT)-1}+\frac{1}{2}\right)$$

[3] Roger H. Koch, D. J. Van Harlingen, and John Clarke. "Measurements of quantum noise in resistively shunted Josephson junctions". In: *Physical Review B* 26.1 (1982).

Application: Planck radiation from dipole emission[4]

An oscillating dipole

$$P = P_0 \sin \omega t$$

with equation of motion

$$m\frac{dv}{dt} + m\omega_0^2 x + F_\mathrm{d} = F \quad \text{with} \quad F_\mathrm{d} = -\frac{2}{3}e^2 c^{-3}\frac{d^2 v}{dt^2}$$

will feel a fluctuating electrical field

$$\langle \mathcal{E}^2 \rangle = \frac{4}{\pi c^3}\int_0^\infty \left(\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \right) \omega^2 d\omega$$

where we can recognize the contribution from Planck radiation.

[4] HB Callen and TA Welton. "Irreversibility and generalized noise". In: *Phys. Rev.* 83 (1951), p. 34.

Review of conditional probabilities

- $W(x_1, t_1; \ldots; x_n, t_n) dx_1 \cdots dx_n$ is the probability to find sample path passing through each of the intervals $(x_i, x_i + dx_i)$ at time t_i
- W is a probability distribution satisfying
  - $W > 0$
  - $\int dx_1 \ldots dx_n \, W = 1$
  - symmetric under permutation index pairs
  - $W(x_1, t_1; \ldots; x_{n-1}, t_{n-1}) = \int dx_n \, W(x_1, t_1; \ldots; x_n, t_n)$
- For a stationary process
  - $W(x_1, t_1) \equiv W(x_1)$
  - $W(x_1, t_1; x_2, t_2) = W(x_1, t_1 - t_2; x_2, 0)$, and so on...
- Bayes' theorem gives the conditional probability:

$$P(x_1, t_1; \ldots; x_k, t_k | x_{k+1}, t_{k+1}; \ldots; x_n, t_n) = \frac{W(x_1, t_1; \ldots; x_n, t_n)}{W(x_1, t_1; \ldots; x_k, t_k)}$$

Markov process, Chapman-Kolmogorov equation

A Markov process is memoryless, i.e. it is characterized by

$$P(x_1, t_1; \dots; x_k, t_k | x_{k+1}, t_{k+1}) = P(x_k, t_k | x_{k+1}, t_{k+1})$$

If it is also stationary, then

$$\begin{aligned} P(x_k, t_k | x_{k+1}, t_{k+1}) &= P(x_k, 0 | x_{k+1}, \Delta t_{k+1}) \\ \text{s.t.} \quad \lim_{\Delta t_{k+1} \to \infty} P(x_k, 0 | x_{k+1}, \Delta t_{k+1}) &= P_s(x_k) \end{aligned}$$

Any Markov process satisfies the consistency relations

- $P(x_1, t_1 | x_3, t_3) = \int dx_2 \, P(x_1, t_1 | x_2, t_2) P(x_2, t_2 | x_3, t_3)$
- $W(x_2, t_2) = \int dx_1 \, W(x_1, t_1) \, P(x_1, t_1 | x_2, t_2)$

Fokker-Planck equation via the Kramers-Moyal expansion

1. Chapman-Kolmogorov equation:

$$P(x_0, t_0|x, t_0 + \Delta t) = \int d\epsilon \, \underbrace{P(x_0, t_0|x - \epsilon, t)P(x - \epsilon, t|x, t + \Delta t)}_{f(x-\epsilon)}$$

2. Taylor expand, $f(x - \epsilon) = f(x) + \sum_{n=1}^{\infty} \frac{(-\epsilon)^n}{n!} \partial_x^n f(x)$
3. Inserting this back, and re-arranging:

$$\frac{P(x_0, t_0|x, t + \Delta t) - P(x_0, t_0|x, t)}{\Delta t} = \sum_n \frac{(-1)^n}{\Delta t \cdot n!} \partial_x^n P(x, t|x_0, t_0) \times \int d\epsilon \, \epsilon^n P(x, t|x + \epsilon, t + \Delta t)$$

4. Defining, $B_n(x, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int \epsilon^n P(\epsilon, \Delta t)$,

$$\partial_t P(x_0, t_0|x, t) = \sum_n \frac{(-1)^n}{n!} \partial_x^n \left[B_n(x, t) P(x_0, t_0|x, t) \right]$$

Fokker-Planck equation for a continuous Markov process

First, we note from the definition that in fact, $B_n(x,t) = \lim_{\Delta t \to 0} \frac{\langle x^n \rangle}{\Delta t}$
Recall, that a continuous Markov process is described by

$$dx = A(x,t)dt + \sqrt{D(x,t)}\,dW$$

we have,

$$\begin{aligned} \langle x \rangle &= A(x,t)dt & \Rightarrow \quad B_1 &= A(x,t) \\ \langle x^2 \rangle &= D(x,t)(\sqrt{dt})^2 & \Rightarrow \quad B_2 &= D(x,t) \\ \langle x^n \rangle &= 0 \text{ for } n > 2 \end{aligned}$$

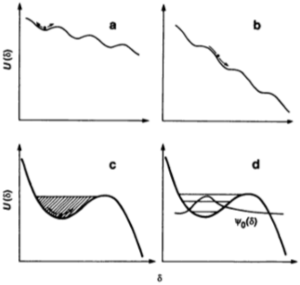
Thus, the corresponding Fokker-Planck equation is,

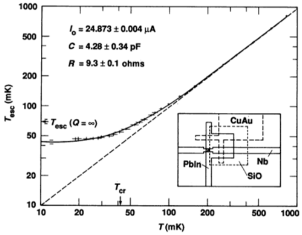
$$\partial_t P = -\partial_x \left[A(x,t)P\right] + \frac{1}{2}\partial_x^2 \left[D(x,t)P\right]$$

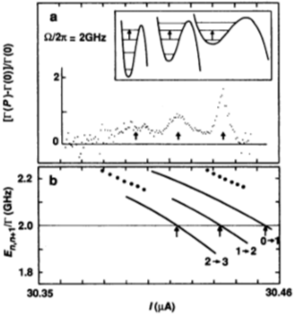
Next week's paper

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

John Clarke, Andrew N. Cleland, Michel H. Devoret, Daniel Esteve, John M. Martinis







Questions for the paper presentation

- What is a Josephson junction? What are the Josphson relations?
- Explain how the Kramers escape rate could be applied to a Josephson junction.
- How does quantum mechanics modify the classical Kramers escape problem?
- How do they measure the energy levels by escape rate? (Explain fig.6)
- How will graph 6(a) change as the temperature increases?