Stat. Phys. IV: Lecture 2

Spring 2025

Deterministic, continuous, memory-less process

Update formula

$X(t + dt) = X(t) + A(X(t), t)dt$

- Same formula with $dt \to \sqrt{dt}$ is not consistent
  - Original update: $X(t + dt) = X(t) + A(X(t), t)dt$ (order $O(dt)$).
  - If $dt \to \sqrt{dt}$:
    $$X(t + dt) = X(t) + A(X(t), t)\sqrt{dt}$$
  - Two-step update:
    $$\begin{aligned} X(t + \frac{dt}{2} + \frac{dt}{2}) &= X(t) + A(X(t), t)\sqrt{\frac{dt}{2}} + A(X(t), t)\sqrt{\frac{dt}{2}} \\ &= X(t) + A(X(t), t) \cdot \sqrt{2} \cdot \sqrt{dt} \end{aligned}$$
  - Inconsistency arises because:
    $$\sqrt{2} \cdot \sqrt{dt} \neq \sqrt{dt}$$
- X can have multiple dimensions (i.e. $(x(t), v(t))$ for a mechanical process)

Review of basic probability theory

- Variance : $\mathrm{var}[X] = \langle X^2 \rangle - \langle X \rangle^2$
- Standard deviation: $\sigma = \sqrt{\mathrm{var}[X]}$
- Covariance: $\mathrm{cov}[X_1, X_2] = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$
- Gaussian random variable: $Y = \mathbf{N}(\mu, \sigma^2)$ with probability density $P(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$
  Properties: $\alpha + \beta \mathbf{N}(\mu, \sigma^2) = \mathbf{N}(\alpha + \beta\mu, \beta^2\sigma^2)$ and
  $\mathbf{N}(\mu_1, \sigma_1) + \mathbf{N}(\mu_2, \sigma_2) = \mathbf{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
  Unit gaussian $\mathbf{N} \equiv \mathbf{N}(0, 1)$

Stochastic continuous Markov process[1]

Update formula (Langevin equation)

$$X(t+dt) = X(t) + \underbrace{A(X(t),t)}_{\text{drift}}\,dt + \underbrace{\sqrt{D(X(t),t)}}_{\text{diffusion}}\,N(t)\sqrt{dt}$$

- **Why it is consistent (though not differentiable):**
  - One-step update:

$$X(t+dt) = X(t) + A(X(t),t)dt + \sqrt{D(X(t),t)}N(t)\sqrt{dt}$$

  - Two-step update:

$$X(t+\frac{dt}{2}) = X(t) + A(X(t),t)\frac{dt}{2} + \sqrt{D(X(t),t)}N_1(t)\sqrt{\frac{dt}{2}}$$

$$X(t+dt) = X(t+\frac{dt}{2}) + A(X(t+\frac{dt}{2}),t+\frac{dt}{2})\frac{dt}{2} + \sqrt{D(X(t+\frac{dt}{2}),t+\frac{dt}{2})}N_2(t)\sqrt{\frac{dt}{2}}$$

  - Since $N_1, N_2 \sim \mathcal{N}(0,1)$ and independent:

$$N_1\sqrt{\frac{dt}{2}} + N_2\sqrt{\frac{dt}{2}} \sim N(t)\sqrt{dt}$$

  - Consistent as both updates give the same distribution.

Ornstein-Uhlenbeck Process

$A(x,t) = -\frac{x}{\tau}$ and $D(x,t) = c$

Update formula

$\frac{dX(t)}{dt} = -\frac{1}{\tau}X(t) + \sqrt{c}N(t)\sqrt{dt}$

Properties (to be proved in coming slides)

- Average: $\langle X(t) \rangle = x_0 e^{-(t-t_0)/\tau}$, $t \geq t_0$
- Covariance: $\text{cov}\left(X(t), X(t')\right) = \frac{c\tau}{2} e^{-(t'-t)/\tau}\left(1 - e^{-2(t-t_0)/\tau}\right)$, $t' \geq t \geq t_0$
- Long time limit: $\lim_{t\to\infty} X(t) = \mathbf{N}(0, c\tau/2)$

Derivations for O-U Process

1. Mean of $X(t)$:

$$dX(t) = -\frac{1}{\tau}X(t)dt + \sqrt{c}N(t)\sqrt{dt}$$
$$\langle dX(t)\rangle = -\frac{1}{\tau}\langle X(t)\rangle dt$$
$$\frac{d\langle X(t)\rangle}{dt} = -\frac{1}{\tau}\langle X(t)\rangle \implies \langle X(t)\rangle = x_0 e^{-(t-t_0)/\tau}$$

2. Second Moment of $X(t)$:

$$dX^2(t) = 2X(t)dX(t) + (dX(t))^2$$
$$= 2X(t)\left(-\frac{1}{\tau}X(t)dt + \sqrt{c}N(t)\sqrt{dt}\right) + cN^2(t)dt$$
$$\frac{d\left\langle X^2(t)\right\rangle}{dt} = -\frac{2}{\tau}\left\langle X^2(t)\right\rangle + c \implies \left\langle X^2(t)\right\rangle = \frac{c\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right)$$

Variance and Long Time Limit for O-U Process

1. **Variance of $X(t)$:**

$$\begin{aligned}\text{Var}(X(t)) &= \left\langle X^2(t)\right\rangle - \left\langle X(t)\right\rangle^2 \\ &= \frac{c\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right) - x_0^2 e^{-2(t-t_0)/\tau}\end{aligned}$$

2. **Long Time Limit:**

$$\lim_{t\to\infty}\langle X(t)\rangle = 0, \quad \lim_{t\to\infty}\text{Var}(X(t)) = \frac{c\tau}{2}$$

$$\Rightarrow X(t) \sim \mathcal{N}\left(0, \frac{c\tau}{2}\right)$$

Example: Brownian motion

Ornstein-Uhlenbeck process for the velocity

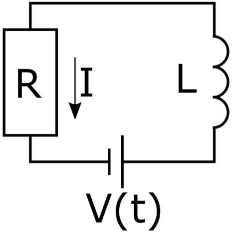
Equation of motion

$$\frac{dv(t)}{dt} = -\frac{1}{\tau}v(t) + \sqrt{c}\Gamma(t), \qquad \Gamma(t), \qquad \Gamma(t) = N(t)/\sqrt{dt}$$

with $\tau = m/\gamma$ and $c = \frac{2}{D}\left(\frac{kT}{m}\right)^2$ (D is the diffusion constant)

Example: RL-circuit[2,3]

Ornstein-Uhlenbeck process for the current



Khirchoff's law

$$L\frac{dI(t)}{dt} = -RI(t) + V(t)$$

- I — current in the circuit
- R — resistance
- L — inductance
- $V(t) = \sqrt{2k_BTR}\Gamma(t)$ — fluctuating voltage, $\langle\Gamma(t)\Gamma(t')\rangle = \delta(t-t')$

The corresponding Ornstein-Uhlenbeck coefficients are $\tau = R/L$ and $c = 2k_BTR/L^2$.

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[2] Daniel T. Gillespie. "The mathematics of Brownian motion and Johnson noise". In: *American Journal of Physics* 64.3 (Mar. 1, 1996), pp. 225–240. DOI: 10.1119/1.18210.

[3] J. Johnson and H. Nyquist. "Thermal agitation of electricity in conductors & Thermal agitation of electricity in conductors". In: *Phys. Rev.* 64 (6, 1928), pp. 97–113.

Application: noise thermometry[4]

Noise voltage variance across a resistance R per Hz is given by

Nyquist's formula

$\langle V^2(\nu) \rangle = 4k_B T R(\nu)$

- One-sided spectra are employed
- $R(\nu) = \mathrm{Re}[Z(\nu)]$ — frequency-dependent resistance

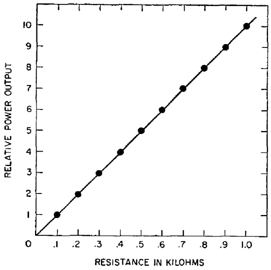

Fig. 3. Amplified noise power output *vs* sensing resistance for a temperature of 293°K.

[4] F. Shore. "Low-Temperature Thermal Noise Thermometer". In: *Review of Scientific Instruments* 30.7 (4, 1959), pp. 578–580.

One-sided Power Spectral Density[5]

$$
\begin{aligned}
&\langle X(t)X(t+\tau)\rangle = C_{XX}(\tau),\\
&C_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}d\omega\\
&S_{XX}(\omega) = \int_{-\infty}^{\infty} C_{XX}(\tau)e^{-i\omega\tau}d\tau = 2\int_{0}^{\infty} C_{XX}(\tau)\cos(\omega\tau)d\tau
\end{aligned}
$$

- $X(t)$ — stochastic process
- $C_{XX}(\tau)$ — auto-correlation function
- $S_{XX}(\omega)$ — power spectral density

*Note that, unlike in classical case, for quantum variables generally there is no one-sided PSD.

[5] Daniel T. Gillespie. "The mathematics of Brownian motion and Johnson noise". In: *American Journal of Physics* 64.3 (Mar. 1, 1996), pp. 225–240. DOI: 10.1119/1.18210.

Derivation of Auto-correlation Function for O-U process: $C_{XX}(\tau)$

- For the O-U process:

$$dX(t) = -\frac{1}{\tau}X(t)dt + \sqrt{c}N(t)\sqrt{dt}$$

- Assume $\langle X(t)\rangle = 0$. Thus, $C_{XX}(\tau) = \langle X(t)X(t+\tau)\rangle$.

Derivation:

$$\langle X(t)X(t+\tau)\rangle = \left\langle X(t)\left(X(t)e^{-\frac{\tau}{\tau}} + \int_0^{\tau}\sqrt{c}e^{-\frac{s}{\tau}}N(t+s)ds\right)\right\rangle$$

$$= \left\langle X^2(t)\right\rangle e^{-\frac{\tau}{\tau}}$$

$$= \frac{c\tau}{2}e^{-\frac{|\tau|}{\tau}}$$

Derivation of Power Spectral Density $S_{XX}(\omega)$

Definition

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} C_{XX}(\tau) e^{-i\omega\tau} d\tau$$

Substitute $C_{XX}(\tau) = \frac{c\mathcal{T}}{2} e^{-\frac{|\tau|}{\mathcal{T}}}$:

$$S_{XX}(\omega) = \frac{c\mathcal{T}}{2} \int_{-\infty}^{\infty} e^{-\frac{|\tau|}{\mathcal{T}}} e^{-i\omega\tau} d\tau$$

Split the Integral:

$$S_{XX}(\omega) = \frac{c\mathcal{T}}{2} \left(\int_{0}^{\infty} e^{-\frac{\tau}{\mathcal{T}}} e^{-i\omega\tau} d\tau + \int_{-\infty}^{0} e^{\frac{\tau}{\mathcal{T}}} e^{-i\omega\tau} d\tau \right)$$

Computing $S_{XX}(\omega)$ - Continued

Compute the integrals:

$$\int_0^{\infty} e^{-\frac{\tau}{\tau}} e^{-i\omega\tau} d\tau = \frac{\tau}{1 + i\omega\tau}$$

$$\int_{-\infty}^{0} e^{\frac{\tau}{\tau}} e^{-i\omega\tau} d\tau = \frac{\tau}{1 - i\omega\tau}$$

Combine results:

$$S_{XX}(\omega) = \frac{c\tau}{2} \left(\frac{\tau}{1 + i\omega\tau} + \frac{\tau}{1 - i\omega\tau} \right)$$

Simplify:

$$S_{XX}(\omega) = \frac{c\tau^2}{1 + (2\pi\omega\tau)^2}$$

One-sided Power Spectral Density $S_{XX}(\omega)$

Definition

$$S_{XX}(\omega) = 2\int_0^\infty C_{XX}(\tau)\cos(\omega\tau)d\tau$$

Using $C_{XX}(\tau) = \frac{c\mathcal{T}}{2}e^{-\frac{|\tau|}{\mathcal{T}}}$:

$$S_{XX}(\omega) = \frac{c\mathcal{T}}{2}\int_0^\infty e^{-\frac{\tau}{\mathcal{T}}}\cos(\omega\tau)d\tau$$

Solve the integral:

$$S_{XX}(\omega) = \frac{2c\mathcal{T}^2}{1+(2\pi\omega\mathcal{T})^2}$$

Interpretation:

- Describes how different frequencies contribute to the variance of $X(t)$.
- Peak at low frequencies indicates long-term correlations, $\sim \omega^{-2}$ for high-frequency: pink noise.

Log-Log Plot of Power Spectral Density for O-U Process

Key Parameters:

- Noise strength $c = 1$
- Time scale $\tau = 1$

Key Observations:

- **Low Frequency:** Power is constant, indicating long-term correlations.
- **High Frequency:**

$$S_{XX}(\omega) \approx \frac{c}{2\pi^2\omega^2}$$

  Decays as $\frac{1}{\omega^2}$.
- **Power:**
  - Shaded area = total power (variance).
  - Consistent with Wiener-Khinchin theorem.

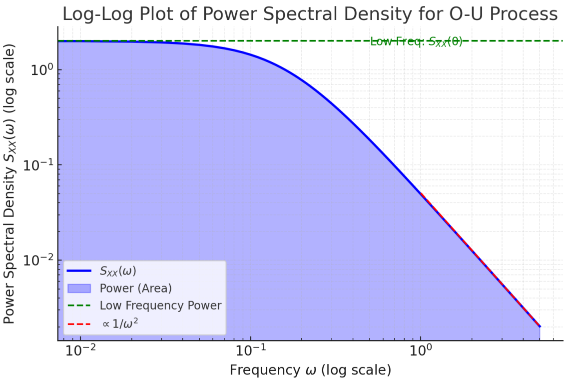

Figure: Log-log plot of $S_{XX}(\omega)$.

Inverse Relation and Wiener-Khinchin Theorem

Inverse Fourier Transform

$$C_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$

Wiener-Khinchin Theorem

$$\langle X^2(t) \rangle = \int_{0}^{\infty} S_{XX}(\omega) d\omega$$

Interpretation:

- Total power (variance) is the area under the spectral density curve.
- Links time-domain variance to frequency-domain description.

An example of PSD

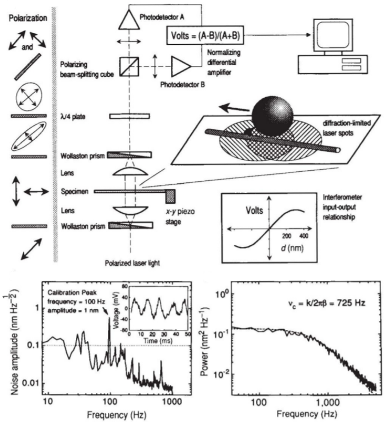

Figure: Depth calibration of an optical trap by Brownian motion PSD of a trapped bead - K. Svoboda et al. "Direct observation of kinesin stepping by optical trapping interferometery". In: *Nature* 365 (1993), pp. 721–727.

Paper for next week's presentation

PHYSICAL REVIEW D

PARTICLES AND FIELDS

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Thermal noise in mechanical experiments

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The fluctuation-dissipation theorem is applied to the case of low-dissipation mechanical oscillators, whose losses are dominated by processes occurring inside the material of which the oscillators are made. In the common case of losses described by a complex spring constant with a constant imaginary part, the thermal noise displacement power spectrum is steeper by one power of ω than is predicted by a velocity-damping model. I construct models for the thermal noise spectra of systems with more than one mode of vibration, and evaluate a model of a specific design of pendulum suspension for the test masses in a gravitational-wave interferometer.

Questions for paper presentation

- What is the Langevin equation for a mechanical resonator?
- How is the PSD of displacement fluctuation calculated?
- What is admittance and how is it derived?
- Explain internal and external damping mechanisms.
- Explain how the equipartition theorem is restored theoretically and experimentally.
- How is the model generalized to multimodes systems and continuous systems?