Stat. Phys. IV: Lecture 12

Spring 2025

Applying the Quantum Master Equation[1]

Energy relaxtion: $\hat{c} = \hat{a}$ and $\hat{c}^\dagger = \hat{a}^\dagger$. We can consider a dephasing interaction ($\hat{c} = \hat{a}^\dagger \hat{a}$):

$$\hat{H}_{\text{int}} = \sum_k \hbar g_k (\hat{a}^\dagger \hat{a} \hat{b}_k^\dagger + \hat{b}_k \hat{a}^\dagger \hat{a}) \ .$$

Defining $\frac{1}{T_\Phi} = 2\pi D(\omega) \left| g(\omega) \right|^2$ as the dephasing rate the QME becomes:

$$\dot{\hat{\rho}} = -i\omega \left[\hat{a}^\dagger \hat{a}, \hat{\rho} \right] + \frac{1}{2} \frac{1}{T_\Phi} (2\bar{n}_m + 1) \left\{ 2\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} - (\hat{a}^\dagger \hat{a})^2 \hat{\rho} - \hat{\rho} (\hat{a}^\dagger \hat{a})^2 \right\} \ .$$

Thus a phase damped oscillator in the Fock basis obeys

$$\dot{\rho}_{nm} = i\omega(n - m) - \frac{1}{2} \frac{1}{T_\Phi} (2\bar{n}_m + 1)(n - m)^2 \rho_{nm}$$

[1] "Quantum Noise", P. Zoller, Chapter 6

QME for Two Level System

$$H_S = \sum_k \hbar g_k(\hat{\sigma}_+ \hat{b}_k + \hat{\sigma}_- \hat{b}_k^\dagger)$$

$$\begin{aligned}\dot{\hat{\rho}} = -i\frac{\omega_s}{2}[\hat{\sigma}_z, \hat{\rho}] \quad &+ \quad \frac{\gamma}{2}(\bar{n}_m + 1)(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-) \\ &+ \quad \frac{\gamma}{2}\bar{n}_m(2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+)\end{aligned}$$

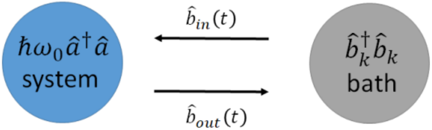
This leads to the equations of motion

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_+\rangle &= -\frac{1}{2}\gamma(2\bar{n}+1)\langle\hat{\sigma}_+\rangle + \underbrace{i\Omega}_{\text{Drive}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_-\rangle &= -\frac{1}{2}\gamma(2\bar{n}+1)\langle\hat{\sigma}_-\rangle - \underbrace{i\Omega^*}_{\text{Drive}} \\ \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle &= -\gamma(2\bar{n}+1)\langle\hat{\sigma}_z\rangle - \gamma + \underbrace{\frac{i}{2}(\Omega^*\langle\hat{\sigma}_+\rangle - \Omega^*\langle\hat{\sigma}_-\rangle)}_{\text{Drive}}\end{aligned}$$

Comments on the results for the two level system

- Stationary solutions are $\langle\hat{\sigma}_+\rangle = \langle\hat{\sigma}_-\rangle = 0$ and $\langle\hat{\sigma}_z\rangle = -\frac{1}{2\bar{n}+1}$, $\bar{n} = (\exp\big(\hbar\omega/k_B T\big) - 1)^{-1}$.
- $\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle = -\gamma(2\bar{n}+1)\langle\hat{\sigma}_z\rangle$, so **relaxation is temperature dependent**. This is different to the harmonic oscillator.
- A zero temperature bath corresponds a decay at the atomic spontaneous emission rate $\gamma = |g_k|^2 D(\omega) 2\pi$
- We can include a drive field to excite the atom $H_{\mathsf{Drive}} = \frac{1}{2}(\hbar\Omega^* \hat{a} + \hbar\Omega\hat{a}^\dagger)$

Quantum reservoir engineering[2]



- Powerful concept to create quantum states (i.e. non-classical states)
- $\dot{\rho}_s = \kappa(2\hat{c}\rho\hat{c}^\dagger - \hat{c}^\dagger\rho\hat{c} - \hat{c}\rho\hat{c}^\dagger)$ for $\bar{n}_{\rm th} = 0$ where $\hat{c}$: jump operator
- $\hat{c} = \hat{a}$: relaxation
- $\hat{c} = \hat{a}^\dagger\hat{a}$: relaxation

[2] Poyatos, J. F., J. I. Cirac, and P. Zoller. "Quantum reservoir engineering with laser cooled trapped ions." Physical review letters 77.23 (1996): 4728.

Quantum reservoir engineering: Pointer Basis

- Concept: "Pointer Basis" (Zurek): The coupling to the environment singles out a preferred set of states, the "pointer basis".
- The system relaxes into the eigenstate of $\hat{c}$ with **zero** eigenvalue, as this state does not evolve with time. e.g. for $\hat{c} = \hat{a}$ the pointer state that system relaxes to is $|0\rangle$

Quantum reservoir engineering: squeezed and cat state

- For $\hat{c} = (\mu\hat{a} + \nu\hat{a}^\dagger)$ the eigenstate is a squeezed state $|\zeta\rangle$
- $\mu^2 - \nu^2 = 1$ e.g. $\mu = \cosh(r)$ and $\nu = \sinh(r)$
- Quadratures are defined as: $\hat{X}_1 = \hat{a} + \hat{a}^\dagger$ and $\hat{X}_2 = \hat{a} - \hat{a}^\dagger$ with the variances: $\langle \Delta\hat{X}_1{}^2\rangle = \frac{1}{2}e^{-2r}$ and $\langle \Delta\hat{X}_2{}^2\rangle = \frac{1}{2}e^{2r}$
- For $\hat{c} = (\hat{a} - \alpha)(\hat{a} - \beta)$ in case of $\alpha = -\beta$ the eigenstate is $|\psi\rangle = (|\alpha\rangle + |-\alpha\rangle)$ so the system relaxes to a Schrodinger cat state.[3]

[3] Poyatos, J. F., J. I. Cirac, and P. Zoller. "Quantum reservoir engineering with laser cooled trapped ions." Physical review letters 77.23 (1996): 4728.

Spectral densities in quantum mechanics[4]

Quantum mechanical definition:

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \left\langle \hat{x}(t)\hat{x}(0) \right\rangle dt$$

where $\hat{x}$ is an operator in the Heisenberg picture.
For a simple harmonic oscillator, the autocorrelation is:

$$G_{xx}(t) = \left\langle \hat{x}(t)\hat{x}(0) \right\rangle = \left\langle \hat{x}^2(0) \right\rangle cos(\omega t) + \frac{1}{m\Omega} \left\langle \hat{p}(0)\hat{x}(0) \right\rangle sin(\omega t)$$

In thermal equilibrium, $\left\langle \hat{x}(0)\hat{p}(0) \right\rangle = - \left\langle \hat{p}(0)\hat{x}(0) \right\rangle = \frac{i\hbar}{2}$

[4] "Introduction to quantum noise, measurement, and amplification", Rev. Mod. Phys (2010)

Spectral densities in quantum mechanics

Thus, even for an hermitian operator like $\hat{x}$, G_{xx} can be complex-valued:

$$G_{xx}(t) = \frac{\hbar}{2m\omega}\left(\bar{n}(\hbar\Omega)\,e^{i\omega t} + \left(\bar{n}(\hbar\Omega)+1\right)e^{-i\omega t}\right)$$

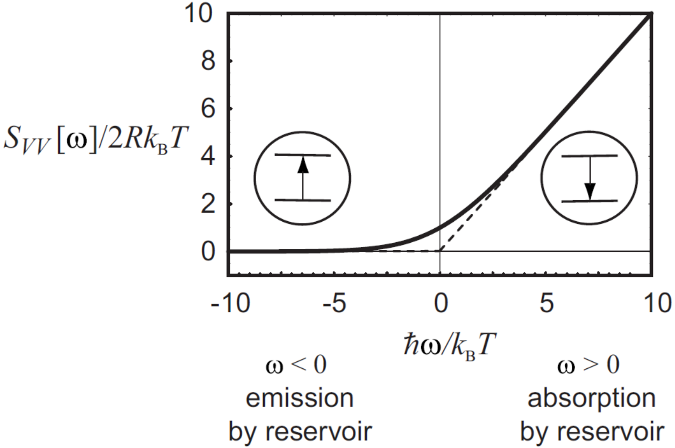
Since the harmonic oscillator autocorrelation is complex, its spectral density is not symmetric in frequency:

$$S_{xx}(\omega) = \frac{\pi\hbar}{m\omega}\left(\bar{n}(\hbar\omega)\delta(\omega+\Omega) + \left(\bar{n}(\hbar\omega)+1\right)\delta(\omega-\Omega)\right)$$

The negative-frequency part is associated to stimulated emission of energy by the oscillator, while the positive-frequency part is related to absorption.

Voltage fluctuations in a resistor[5]

Example: voltage fluctuations in a resistor near absolute zero.



[5]"Introduction to quantum noise, measurement, and amplification", Rev. Mod. Phys (2010)

Paper: Quantum Reservoir Engineering[6]

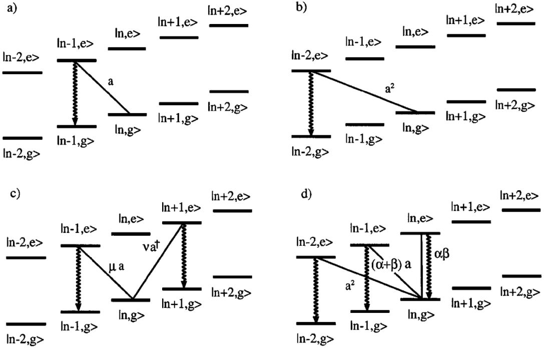


Figure: Laser configurations for several coupling operators f. (a) Laser tuned to $|n, g\rangle \rightarrow |n-1, e\rangle$,which rapidly decays into the state $|n-1, g\rangle$,leading to $f = a$; (b)$f = a^2$; (c)$f = \mu a + \nu a^\dagger$;(d) $f = (a-\alpha)(a-\beta)$.

[6]Poyatos, J. F., J. I. Cirac, and P. Zoller. "Quantum reservoir engineering with laser cooled trapped ions." Physical review letters 77.23 (1996): 4728.