Stat. Phys. IV: Lecture 11

Spring 2025

Quantum master equation for a harmonic oscillator

We can treat the problem of a harmonic oscillator interacting with a heat bath by using the density matrix formalism [1].

System, bath and interaction Hamiltonians:

$$\hat{H}_S = \hbar\omega_0 \hat{a}^\dagger \hat{a},\ \hat{H}_B = \sum_k \hbar\omega_k \hat{b}^\dagger \hat{b},\ \hat{H}_{SB} = \sum_k \hbar\, g_k \hat{a}\hat{b}_k^\dagger + h.c.$$

ρ_{SB} is the density matrix of the interacting system. In the interaction picture

$$\tilde{\rho}_{SB} = e^{i\frac{(\hat{H}_S+\hat{H}_B)t}{\hbar}} \rho_{SB} e^{-i\frac{(\hat{H}_S+\hat{H}_B)t}{\hbar}},$$

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_{SB} = \left[\tilde{\hat{H}}_{SB}(t), \tilde{\rho}_{SB}\right],$$

where $\tilde{\hat{H}}_{SB}$ is the interaction Hamiltonian in the interaction picture.
Integrating this equation formally and substituting back into the previous equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_{SB} = \frac{1}{i\hbar}\left[\tilde{\hat{H}}_{SB}(t), \tilde{\rho}_{SB}(0)\right] + \frac{1}{(i\hbar)^2}\int_0^t \mathrm{d}t' \left[\tilde{\hat{H}}_{SB}(t), \left[\tilde{\hat{H}}_{SB}(t'), \tilde{\rho}_{SB}(t')\right]\right].$$

[1] Charmicheal, "Stochastic Methods in Quantum Optics"

Quantum master equation for a harmonic oscillator (2)

The system density matrix is $\mathrm{tr}_B\{\rho_{SB}\}$, and we assume that $\mathrm{tr}_B\left\{\left[\hat{\tilde{H}}_{SB}(t), \rho_S(0)\right]\right\} = 0$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_S(t) = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{tr}_B\left\{\left[\hat{\tilde{H}}_{SB}(t), \left[\hat{\tilde{H}}_{SB}(t'), \tilde{\rho}_{SB}(t')\right]\right]\right\}$$

Because the bath is large, its state is not, to the first order, affected by the system so $\tilde{\rho}_{SB} = \tilde{\rho}_S \otimes \rho_B(0) + O(\hat{H}_{SB})$. In the Born approximation we neglect the terms $O(\hat{H}_{SB})$ and we use the Markov approximation $(\tilde{\rho}_S(t') \to \tilde{\rho}_S(t))$ to obtain

Master equation in the Born-Markov approximation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_S = -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{tr}_B\left\{\left[\hat{\tilde{H}}_{SB}(t), \left[\hat{\tilde{H}}_{SB}(t'), \tilde{\rho}_S(t)\otimes\rho_B(0)\right]\right]\right\}$$

The system becomes memoryless: it does not depend on the time $t' < t$.

Example of interaction Hamiltonian

Consider $\hat{H}_{SB} = \hbar \sum_k g_k \hat{a}\hat{b}_k^\dagger + h.c.$. We know:

$$\begin{aligned} e^{i\omega_0 \hat{a}^\dagger \hat{a} t} \hat{a} e^{-i\omega_0 \hat{a}^\dagger \hat{a} t} &= \hat{a} e^{-i\omega_0 t} \\ e^{i\sum_k \omega_k \hat{b}_k^\dagger \hat{b} t} \left(\sum_k \hbar g_k \hat{b}_k\right) e^{-i\sum_k \omega_k \hat{b}_k^\dagger \hat{b} t} &= \sum_k \hbar g_k \hat{b}_k e^{-i\omega_k t} \end{aligned}$$

Thus

$$\hat{\tilde{H}}_{SB}(t) = \sum_k \hbar g_k \left(\hat{a}^\dagger \hat{b}_k e^{i(w-w_k)t} + \hat{a}\hat{b}_k^\dagger e^{-i(w-w_k)t} \right) \ .$$

Introducing the bath noise operator $\tilde{\Gamma} = \sum_k g_k \hat{b}_k e^{-i(w-w_k)t}$ we can rewrite the Hamiltonian as $\hat{\tilde{H}}_{SB} = \hbar \left(\hat{a}\tilde{\Gamma}^\dagger + \hat{a}^\dagger \tilde{\Gamma} \right)$. In the most general case

$$\hat{\tilde{H}}_{SB} = \hbar \sum_i \tilde{s}_i(t) \tilde{\Gamma}_i(t) \ ,$$

where $\tilde{s}_i$ and $\tilde{\Gamma}_i$ are generalised system and bath operators respectively.

Master equation derivation

With the generalised operators the master equation becomes:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_S &= -\sum_{i,j}\int_0^t dt'\,\mathrm{tr}_B\left\{\left[\tilde{s}_i\tilde{\Gamma}_i,\left[\tilde{s}_j\tilde{\Gamma}_j,\tilde{\rho}_S(t)\otimes\rho_B(0)\right]\right]\right\} \\
&= -\sum_{i,j}\int_0^t dt'\left\{\left[\tilde{s}_i(t)\tilde{s}_j(t')\tilde{\rho}_S(t')-\tilde{s}_j(t')\tilde{\rho}_S(t')\tilde{s}_i(t)\right]\left\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right\rangle_B\right. \\
&\quad\left.+\left[\tilde{\rho}_S(t')\tilde{s}_j(t')\tilde{s}_i(t)-\tilde{s}_i(t)\tilde{\rho}_S(t')\tilde{s}_j(t')\right]\left\langle\tilde{\Gamma}_j(t')\tilde{\Gamma}_i(t)\right\rangle_B\right\}
\end{aligned}
$$

where $\left\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right\rangle_B = \mathrm{tr}_B\left\{\rho_B\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right\}$. To simplify some of the 16 terms above we must assume that $\left\langle\tilde{\Gamma}_i(t)\tilde{\Gamma}_j(t')\right\rangle_{B;i\neq j} \propto \delta(t-t')$.

Environment correlation functions

By assuming the bath to be in a thermal state and take the limit of continuous density of states one can compute the correlation function for the bath operator

$$
\begin{aligned}
\left\langle \hat{\Gamma}(t)\hat{\Gamma}^\dagger(t')\right\rangle_B &= \sum_{k,l} g_k g_l e^{i(\omega_l t' - \omega_k t)}\,\mathrm{tr}_B\left\{\rho_B \hat{b}_k \hat{b}^\dagger_{k'}\right\} \\
&= \sum_k |g_k|^2 e^{-i\omega_k(t-t')}(\bar{n}(\omega_k, T) + 1) \\
&\approx 2\pi \big|g(\omega_s)\big|^2 (\bar{n}+1) D(\omega)\delta(t-t') \\
&= \gamma(\bar{n}+1)\delta(t-t')
\end{aligned}
$$

Where we assumed g_k to be approximately frequency independent.

Quantum Master Equation

Simplifying all the 16 terms and transforming back to the Schrödinger picture we obtain

Lindblad form of the Quantum Master Equation

$$\dot{\rho} = -i\omega\left[\hat{a}^\dagger\hat{a}, \rho\right] + \frac{\gamma}{2}(\bar{n}+1)(2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}(2\hat{a}^\dagger\rho\hat{a} - \hat{a}\hat{a}^\dagger\rho - \rho\hat{a}\hat{a}^\dagger)$$

Or ordering the terms differently

Quantum Master Equation

$$\dot{\rho} = -i\omega\left[\hat{a}^\dagger\hat{a}, \rho\right] + \frac{\gamma}{2}\left(\left[\hat{a}, \rho\hat{a}^\dagger\right] + \left[\hat{a}\rho, \hat{a}^\dagger\right]\right) + \frac{\gamma}{2}\bar{n}\left(\left[\hat{a}\rho, \hat{a}^\dagger\right] + \left[\hat{a}^\dagger, \rho\hat{a}\right]\right)$$

It should be noted that in the Lindblad form, the operators $\hat{a}$ can be replaced by operators $\hat{c}$ which are any generalised system operators entering the system-bath coupling.

Other forms of the QME

We can write the QME using the Lindblad operators

QME - Lindblad operators

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] + \sum_k \left(\hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2}\left\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}\right\}\right)$$

For the example above $\hat{L}_{1,-} = \sqrt{\gamma(\bar{n}+1)}\hat{a}$ and $\hat{L}_{1,+} = \sqrt{\gamma\bar{n}}\hat{a}^\dagger$.
At $T = 0$, $\bar{n} = 0$ and the QME takes the form

Zero Temperature form of the Quantum Master Equation

$$\dot{\rho} = -i\omega\left[\hat{a}^\dagger \hat{a}, \rho\right] + \frac{\gamma}{2}(2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a})$$

Physical interpretation of the QME

The physical interpretation of the QME follows from the rate equations satisfied by the populations $p_n = \langle n| \rho |n\rangle$, where $|n\rangle$ is the n-th energy eigenstate of the harmonic oscillator.

$$\dot{p}_n = \gamma(\bar{n}+1)(n+1)p_{n+1} - \gamma\bar{n}np_n + \gamma\bar{n}np_{n-1} - \gamma\bar{n}(n+1)p_n$$

That is a birth-death process equation (similar to classical cases). Thus

$$\textit{gain: } g = \frac{\gamma}{2}(\bar{n}+1)n$$
$$\textit{loss: } r = \frac{\gamma}{2}\bar{n}(n+1)$$

QME and P-representation[2]

Glauber–Sudarshan P representation:

$$\hat{\rho} = \int d^2\alpha P(\alpha, \alpha^*) |\alpha\rangle \langle\alpha|$$

This representation can be used to convert the quantum optical master equation to a Fokker-Planck equation.

QME in P-representation

$$\frac{\partial}{\partial t} P(\alpha, \alpha^*, t) = \frac{\gamma}{2}[\frac{\partial}{\partial \alpha}\alpha + \frac{\partial}{\partial \alpha^*}\alpha^*]P(\alpha, \alpha^*, t) + \gamma\bar{n}[\frac{\partial^2 P(\alpha, \alpha^*, t)}{\partial\alpha\partial\alpha^*}]$$

[2] Scully Quantum optics chapter 8

Applying the Quantum Master Equation[3]
Dephasing:

Energy relaxtion: $\hat{c} = \hat{a}$ and $\hat{c}^\dagger = \hat{a}^\dagger$. We can consider a dephasing interaction ($\hat{c} = \hat{a}^\dagger \hat{a}$):

$$\hat{H}_{\text{int}} = \sum_k \hbar g_k (\hat{a}^\dagger \hat{a} \hat{b}_k^\dagger + \hat{b}_k \hat{a}^\dagger \hat{a}) \ .$$

Defining $\frac{1}{T_\Phi} = 2\pi D(\omega)\left|g(\omega)\right|^2$ as the dephasing rate the QME becomes:

$$\dot{\hat{\rho}} = -i\omega\left[\hat{a}^\dagger \hat{a}, \hat{\rho}\right] + \frac{1}{2}\frac{1}{T_\Phi}(2\bar{n}_m + 1)\left\{2\hat{a}^\dagger \hat{a} \hat{\rho} \hat{a}^\dagger \hat{a} - (\hat{a}^\dagger \hat{a})^2 \hat{\rho} - \hat{\rho}(\hat{a}^\dagger \hat{a})^2\right\} \ .$$

Thus a phase damped oscillator in the Fock basis obeys

$$\dot{\rho}_{nm} = i\omega(n-m) - \frac{1}{2}\frac{1}{T_\Phi}(2\bar{n}_m + 1)(n-m)^2 \rho_{nm}$$

[3] "Quantum Noise", P. Zoller, Chapter 6

QME for a Two Level System

$$H_S = \sum_k \hbar g_k (\hat{\sigma}_+ \hat{b}_k + \hat{\sigma}_- \hat{b}_k^\dagger)$$

$$\begin{aligned}\dot{\hat{\rho}} = -i\frac{\omega_s}{2}[\hat{\sigma}_z, \hat{\rho}] \quad &+ \quad \frac{\gamma}{2}(\bar{n}_m + 1)(2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ - \hat{\sigma}_+\hat{\sigma}_-\hat{\rho} - \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-) \\ &+ \quad \frac{\gamma}{2}\bar{n}_m(2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+\hat{\rho} - \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+)\end{aligned}$$

This leads to the equations of motion

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_+\rangle &= -\frac{1}{2}\gamma(2\bar{n}+1)\langle\hat{\sigma}_+\rangle + \underbrace{i\Omega\langle\hat{\sigma_z}\rangle}_{\text{Drive}} \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_-\rangle &= -\frac{1}{2}\gamma(2\bar{n}+1)\langle\hat{\sigma}_-\rangle - \underbrace{i\Omega^*\langle\hat{\sigma_z}\rangle}_{\text{Drive}} \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle &= -\gamma(2\bar{n}+1)\langle\hat{\sigma_z}\rangle - \gamma + \underbrace{\frac{i}{2}(\Omega^*\langle\hat{\sigma}_+\rangle - \Omega^*\langle\hat{\sigma}_-\rangle)}_{\text{Drive}}
\end{aligned}$$

Comments on the results for the two level system

- Stationary solutions are $\langle\hat{\sigma}_+\rangle = \langle\hat{\sigma}_-\rangle = 0$ and $\langle\hat{\sigma}_z\rangle = -\frac{1}{2\bar{n}+1}$, $\bar{n} = (\exp\big(\hbar\omega/k_B T\big) - 1)^{-1}$.
- $\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\sigma}_z\rangle = -\gamma(2\bar{n}+1)\langle\hat{\sigma_z}\rangle$, so **relaxation is temperature dependent**. This is different to the harmonic oscillator.
- A zero temperature bath corresponds a decay at the atomic spontaneous emission rate $\gamma = |g_k|^2 D(\omega) 2\pi$
- We can include a drive field to excite the atom $H_{\text{Drive}} = \frac{1}{2}(\hbar\Omega^*\hat{a} + \hbar\Omega\hat{a}^\dagger)$

Qutip: Open quantum systems simulator

Damping of a quantized harmonic oscillator

Consider a quantized free harmonic oscillator, described by the hamiltonian, $H = \hbar\Omega_m a^\dagger a$. This hamiltonian can be constructed in Qutip as follows:

In [1]:
```
# import required packages
from qutip import *
from scipy import *
from matplotlib.pylab import *

# show all plots "inline"
%matplotlib inline

dim=30; # dimension of hilbert space
a=destroy(dim); # annihilation operator in this Hilbert space
psi0=basis(dim,0); # ground state of the oscillator

omegam = 2*pi*1e6; # oscillator frequency
H = omegam*a.dag()*a; # hamiltonian
```

However, any observable oscillator couples to an environment. In timescales which are long compared to the correlation times of the environment variables, this system-environment interaction is described by a Lindblad master equation for the system density matrix:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_k L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}$$

where H is the hamiltonian, and $\{L_k\}_{k=1,\dots,n}$ are jump operators that describe the action of the environment on a pure state of the system.

Here we consider an environment constituted by a large collection of harmonic oscillators which are in thermal equilibrium at a temperature T, and which couple to the system at a rate Γ_m. The relevant jump operators in this case are,

$L_1 = \sqrt{\Gamma_m n_{th}}\, a$, corresponding to emission of a quanta by the system

$L_2 = \sqrt{\Gamma_m (n_{th}+1)}\, a^\dagger$, correponding to addition of thermal quanta into the system

In [2]:
```
# define rates and jump operators
nth = 10;
gammam = 2*pi*1;
```

Figure: Sample simulation code on Qutip

Qutip: Open quantum systems simulator

In [13]:
```
a # Matrix representation of a
```

Out[13]: Quantum object: dims = [[30], [30]], shape = (30, 30), type = oper, isherm = False

$$\begin{pmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.414 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.732 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 2.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 5.099 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 5.196 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 5.292 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 5.385 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

In [14]:
```
commutator(a, a.dag())    # commutator of a and a dagger
```

Out[14]: Quantum object: dims = [[30], [30]], shape = (30, 30), type = oper, isherm = True

$$\begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.000 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 1.000 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 1.000 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & -29.000 \end{pmatrix}$$

Figure: a and $[a, a^\dagger]$ matrix representations

Supplementary to lecture 12

Observation of quantum jumps[4]

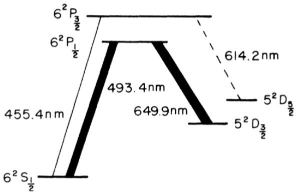


Figure: Level structure of Ba^+. The shelf level is the $5^2D_{5/2}$ state.

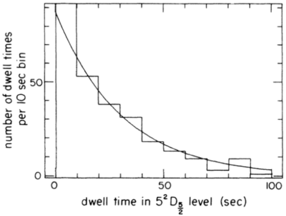


Figure: Histogram showing distribution of dwell times

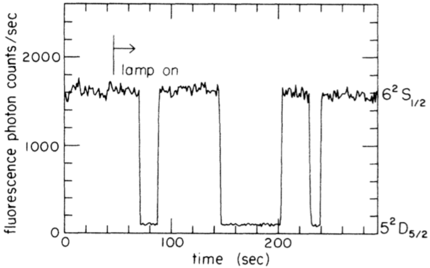


Figure: A typical trace of the 493-nm fluorescence from the $6^2P_{1/2}$ level showing the quantum jumps after the hollow cathode lamp is turned on.

Birth and death of a photon in a cavity [5]

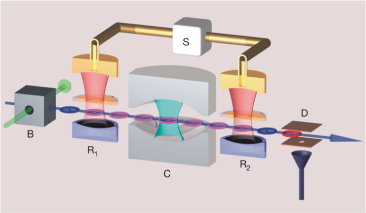
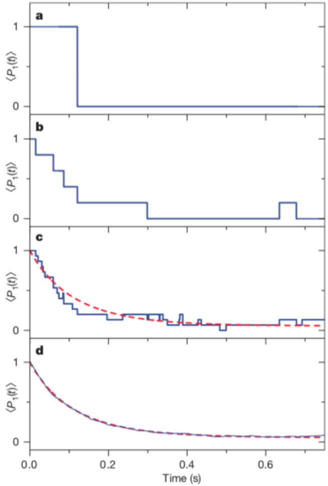


Figure: Experimental set-up

Decay of the one-photon state. a, Measured value of $P_1 = |1\rangle\langle 1|$ as a function of time, in a single experimental realization; b–d, averages of 5, 15 and 904 similar quantum trajectories, showing the gradual transition from quantum randomness into a smooth ensemble average.



[5] Gleyzes et. al. Quantum jumps of light recording the birth and death of a photon in a cavity, Nature volume 446, pages 297–300