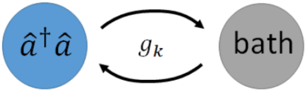
# Stat. Phys. IV: Lecture 10

Spring 2025

# Reminder about the Quantum Langevin Equation

We derived the
quantum Langevin equation:

$$\partial_t \hat{a} = -\frac{\kappa}{2}\hat{a} + i\omega_0\hat{a} + \hat{f}(t)$$



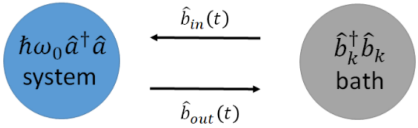

We found the force correlations to be:

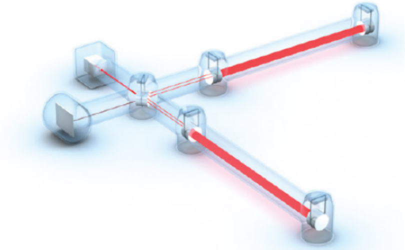
$$\begin{aligned} \left\langle \hat{f}(t)\hat{f}^\dagger(t') \right\rangle &= \kappa(\bar{n}(\omega)+1)\delta(t-t')\ , \\ \left\langle \hat{f}^\dagger(t)\hat{f}(t') \right\rangle &= \kappa(\bar{n}(\omega))\delta(t-t')\ , \end{aligned}$$

where $\kappa = 2\pi D(\omega_0)\left|g(\omega_0)\right|^2$ and $\bar{n}(\omega) = \left\langle \hat{b}_k^\dagger(0)\hat{b}_k(0) \right\rangle$. Note that the second of the above equations yields a manifestation of the fluctuation dissipation theorem:

$$\kappa = \frac{1}{\bar{n}(\omega)} \int \left\langle \hat{f}^\dagger(t)\hat{f}(t') \right\rangle \mathrm{d}t'$$

# Motivation for Input-Output formalism - LIGO



# Input and Output Theory [1]

Total Hamiltonian of an open quantum system:

$$\begin{aligned}
\hat{H} &= \hat{H}_{\text{sys}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{int}} \\
\hat{H}_{\text{sys}} &= \hbar\omega_0 \hat{a}^\dagger \hat{a} \\
\hat{H}_{\text{bath}} &= \hbar \int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega \\
\hat{H}_{\text{int}} &= i\hbar \int_{-\infty}^{\infty} \mathrm{d}\omega\, \kappa(\omega) \cdot \left[\hat{b}^\dagger(\omega)\hat{c} - \hat{c}^\dagger \hat{b}(\omega)\right]
\end{aligned}$$

Use the Heisenberg EOM to find:

$$\partial_t \hat{b}(\omega) = -i\omega \hat{b}(\omega) + \kappa(\omega)\hat{c}(t) \ .$$

Integrate to find the bath modes:

$$\hat{b}(\omega) = \hat{b}_0(\omega) e^{-i\omega(t-t_0)} + \kappa(\omega) \int_{t_0}^{t} e^{-i\omega(t-t')} \hat{c}(t')\, \mathrm{d}t' \ . \tag{1}$$

Which yields the forward quantum langevin equation.

## Forward Quantum Langevin Equation

$$\partial_t \hat{a} = -i\omega_0 \hat{a} - \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\hat{b}_{\text{in}}(t) \ , \ \hat{b}_{\text{in}}(t) = \int \mathrm{d}\omega\, e^{-i\omega(t-t_0)} \hat{b}_0(\omega)$$

[1] See Gardiner and Collett, Phys. Rev. A **31** 3761 (1985).

# Alternative form of the QLE

Use (1) to find:

$$\int \hat{b}(\omega)\,\mathrm{d}\omega = \underbrace{\int \mathrm{d}\omega\,\hat{b}_0(\omega)e^{i\omega(t-t_0)}}_{\hat{b}_{\text{in}}(t)} + \underbrace{\kappa(\omega)\int\int_{t_0}^{t} e^{i\omega(t-t')}\hat{c}(t')\,\mathrm{d}t\,\mathrm{d}\omega}_{\frac{\hat{c}(t)}{2}\sqrt{\gamma}}\ .$$

Thus

$$\int \hat{b}(\omega)\,\mathrm{d}\omega = \hat{b}_{\text{in}}(t) + \frac{\hat{c}(t)}{2}\sqrt{\gamma}\ . \tag{2}$$

# Time reversed Langevin equations

We integrate the bath modes from $t$ to $t_1$ (note: $\hat{b}_0(\omega) = \hat{b}(\omega)_{t=t_0}$ and $\hat{b}_1(\omega) = \hat{b}(\omega)_{t=t_1}$):

$$\hat{b}(\omega) = e^{-i\omega(t-t_1)}\hat{b}_1(\omega) - \kappa(\omega)\int_t^{t_1} e^{-i\omega(t-t')}\hat{c}(t')\,\mathrm{d}t' \ .$$

Solving for the case where $\hat{c} = \hat{a}$ (energy relaxation) yields the **backward quantum langevin equation** ($\hat{b}_{\text{out}}(t) = \int \mathrm{d}\omega\, e^{-i\omega(t-t_1)}\hat{b}_1(\omega)$):

$$\partial_t \hat{a}(t) = -i\omega_0\hat{a} + \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\hat{b}_{\text{out}}(t) \ ,$$

To quickly go from the backward QLE to the forward QLE we substitute $\hat{b}_{\text{in}}(t) \rightarrow \hat{b}_{\text{out}}(t)$ , $\sqrt{\gamma} \rightarrow \sqrt{\gamma}$ , $\frac{\gamma}{2}\hat{a} \rightarrow -\frac{\gamma}{2}\hat{a}$. Hence:

$$\int \hat{b}(\omega)\,\mathrm{d}\omega = \hat{b}_{\text{out}}(t) + \frac{\hat{c}(t)}{2}\sqrt{\gamma} \ .$$

Substracting (2) from the equation above we obtain:

## Input-Output Relation

$$\hat{b}_{\text{out}}(t) - \hat{b}_{\text{in}}(t) = \sqrt{\gamma}\hat{a}$$

# Quantum Langevin Equations for a two level system (1)

Pauli matrices:

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Let's assume our two level system is coupled to an electromagnetic field with infinitely many modes:

$$\hat{H}_{\text{sys}} = \frac{1}{2}\hbar\Omega\hat{\sigma}_z \ , \ \hat{H}_{\text{bath}} = \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k \ ,$$

$$\hat{H}_{\text{int}} = \sum_k \hbar g_k \left( \hat{\sigma}_- \hat{b}_k^\dagger + \hat{\sigma}_+ \hat{b}_k \right)$$

Using the same procedure as previously (c.f. homework) we obtain:

$$\begin{aligned}
\partial_t \hat{\sigma}_- &= -i\Omega\hat{\sigma}_- - \frac{\kappa}{2}\hat{\sigma}_- + \sqrt{\kappa}\hat{\sigma}_z \hat{b}_{\text{in}}(t) \\
\partial_t \hat{\sigma}_+ &= i\Omega\hat{\sigma}_+ - \frac{\kappa}{2}\hat{\sigma}_+ + \sqrt{\kappa}\hat{\sigma}_z \hat{b}_{\text{in}}(t) \\
\partial_t \hat{\sigma}_z &= -\kappa(1 + \hat{\sigma}_z) - 2\sqrt{\kappa}\left( \hat{\sigma}_+ \hat{b}_{\text{in}}(t) + \hat{b}_{\text{in}}^\dagger(t) \hat{\sigma}_- \right)
\end{aligned}$$

# Quantum Langevin Equations for a two level system (2)

The above equations are not closed and simplify significantly if we assume that the atom is initially in the excited state $|2\rangle$ and that the field is in the vacuum state $\left\langle \hat{b}_k^\dagger(0)\hat{b}_k(0)\right\rangle = 0$ at time $t = t_0$. Then

$$\begin{aligned}
\langle \partial_t \hat{\sigma}_- \rangle &= \left(-i\Omega - \frac{\kappa}{2}\right)\langle \hat{\sigma}_- \rangle \ , \\
\langle \partial_t \hat{\sigma}_+ \rangle &= \left(i\Omega - \frac{\kappa}{2}\right)\langle \hat{\sigma}_+ \rangle \ , \\
\langle \partial_t \hat{\sigma}_z \rangle &= -\kappa\left(1 + \langle \hat{\sigma}_z \rangle\right) \ ,
\end{aligned}$$

where we have assumed $\left\langle \hat{\sigma}_z \hat{b}_{\text{in}}(t)\right\rangle = \langle \hat{\sigma}_z \rangle \left\langle \hat{b}_{\text{in}}(t)\right\rangle$ (i.e. the system is in a separable state). Hence the atom decays as:

$$\left\langle \hat{\sigma}_z(t)\right\rangle = \hat{\sigma}_z(0)e^{-\kappa t}, \kappa = 2\pi\left|g_k\right|^2 D(\omega) \ .$$

Here $D(\omega) = \frac{\omega^2}{c^3\pi^2}$ is the density of states of the electromagnetic field. We obtain a rigorous expression for the atomic decay $\kappa = \frac{\omega^3 e^2 \langle 1|\vec{r}|2\rangle}{3\pi\epsilon_0 c^3}$.

# Nobel Prize 2012 - Serge Haroche

$$
\begin{aligned}
\hat{H}_{\text{cav}} &= \hbar\omega_c \hat{a}^\dagger \hat{a} \\
\hat{H}_{\text{atom}} &= \frac{1}{2}\hbar\Omega\hat{\sigma}_z \\
\hat{H}_{\text{bath}} &= \hbar\sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k \\
\hat{H}_{\text{int}}^{\text{atom}} &= \hbar g(\sigma^+\hat{a} + \sigma^-\hat{a}^\dagger) \\
\hat{H}_{\text{int}}^{\text{cav}} &= \hbar\sum_k g_k^c(\hat{a}\hat{b}_k^\dagger + \hat{a}^\dagger\hat{b}_k)
\end{aligned}
$$

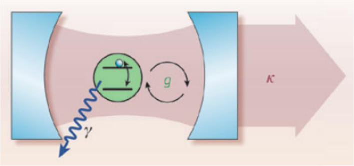



Figure: R.J. Schoelkopf, S.M. Girvin, Nature **451**, 664-669 (2008)

We can consider a simple case where $\hbar\omega \gg k_{\text{B}}T$ (i.e. $\bar{n}_{\text{th}} \simeq 0$). Then, under the weak coupling condition $g \ll \kappa$:

$$\langle \hat{\sigma}_z(t) \rangle = -1 + 2e^{-4g^2t/\kappa} .$$

The atomic decay rate is thus $\Gamma_c = \frac{4g^2}{\kappa}$. This enhancement of the spontaneous emission rate (i.e. it is faster than in free space) is called the Purcell Effect (Phys. Rev. **69**, 37 (1946)).

# Quantum regression theorem[2]

Useful to compute correlations of operators $\left\langle \hat{A}(t)\hat{B}(t+\tau) \right\rangle$

For a complete basis of operators $\left\{ \hat{A}_\mu \right\}$, in the sense that $\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \hat{A}_\mu(t) \right\rangle = \sum_\nu M_{\mu\nu} \left\langle \hat{A}_\nu(t) \right\rangle$, we have

## Quantum regression theorem

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left\langle \hat{O}(t)\hat{A}_\mu(t+\tau) \right\rangle = \sum_\nu M_{\mu\nu} \left\langle \hat{O}(t)\hat{A}_\nu(t+\tau) \right\rangle$$

for any system operator $\hat{O}$.

Example: (1st order correlation function), with $O = \hat{a}^\dagger$, $\hat{A} = \hat{a}$

$$\left\langle \hat{a}^\dagger(t)\hat{a}(t+\tau) \right\rangle = \left\langle \hat{n}(t) \right\rangle e^{-\frac{\gamma}{2}\tau - i\omega_0\tau}$$

[2] H. J. Carmichael, "Statistical Methods in Quantum Optics", section 1.5

# Quantum regression theorem

Can be extended to 3 operators:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left\langle \hat{O}_1(t)\hat{A}_\mu(t+\tau)\hat{O}_2(t)\right\rangle = \sum_\nu M_{\mu\nu}\left\langle \hat{O}_1(t)\hat{A}_\nu(t+\tau)\hat{O}_2(t)\right\rangle$$

Example: (2nd order correlation function)

**Hanbury-Brown-Twiss Effect**

$$\left\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\right\rangle = \bar{n}_e^2\left(1+e^{-\gamma t}\right)$$

It is an example of photon bunching.

## A NEW METHOD FOR THE DETECTION OF A PERIODIC SIGNAL OF UNKNOWN SHAPE AND PERIOD

P. C. GREGORY

Department of Physics, University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia, Canada V6T 1Z1

AND

THOMAS J. LOREDO

Department of Astronomy, Space Sciences Building, Cornell University, Ithaca, NY 14853

*Received 1992 January 6; accepted 1992 April 20*

### ABSTRACT

We present a new method for the detection and measurement of a periodic signal in a data set when we have no prior knowledge of the existence of such a signal or of its characteristics. It is applicable to data consisting of the locations or times of discrete events. We use Bayes's theorem to address both the signal detection problem and the estimation problem of measuring the characteristics of a detected signal. To address the detection problem, we use Bayes's theorem to compare a constant rate model for the signal to models with periodic structure. The periodic models describe the signal plus background rate as a stepwise distribution in $m$ bins per period, for various values of $m$. The Bayesian posterior probability for a periodic model contains a term which quantifies Ockham's razor, penalizing successively more complicated periodic models for their greater complexity even though they are assigned equal prior probabilities. The calculation thus balances model simplicity with goodness of fit, allowing us to determine both whether there is evidence for a periodic signal, and the optimum number of bins for describing the structure in the data. Unlike the results of traditional "frequentist" calculations, the outcome of the Bayesian calculation does not depend on the number of periods examined, but only on the range examined. Once a signal is detected, we again use Bayes's theorem to estimate various parameters of the signal, such as its frequency or the shape of the light curve. The probability density for the frequency is inversely proportional to the multiplicity of the binned events, which is simply related both to the combinatorial entropy of the binned distribution and to the $\chi^2$ measure of its misfit to a uniform distribution used in the "epoch folding" method for period detection. The probability density for the light-curve shape produces light-curve estimates that are superpositions of stepwise distributions with various phases and number of bins, and which are thus smoother than a simple histogram. Error bars for the light-curve shape are also easily calculated. The method also handles gaps in the data due to intermittent observing or dead time. We apply the method to simulated data generated with both stepwise and sinusoidal light curves and demonstrate that it can sensitively detect such signals and accurately estimate both the signal frequency and its shape, even when the light curve does not have a stepwise shape. We also describe a test for nonperiodic source variability that is a simple modification of our period detection procedure.

*Subject headings:* methods: analytical — methods: numerical