Measurement of the Instantaneous Velocity of a Brownian Particle

Xiaoyi Cheng

27.02.2025

Overview

Experimental measurement of the instantaneous velocity of Brownian particles using the optical tweezers platform:

- Observation of the ballistic regime in Brownian motion.
- Verification of the equipartition theorem and the Maxwell-Boltzmann distribution.

Main references:

1. Li, Tongcang, et al. Measurement of the instantaneous velocity of a Brownian particle. *Science* 328.5986 (2010): 1673-1675.
2. Li, Tongcang. Fundamental tests of physics with optically trapped microspheres. *Springer Science & Business Media*, 2012.

Basics: Maxwell-Boltzmann Distribution

Assume that the particles do not interact (classical). In thermal equilibrium, the probability for the particle to be found in a single-particle microstate of energy E_i is

$$P(E_i) = \frac{\exp\left(-\frac{E_i}{k_B T}\right)}{\sum_j \exp\left(-\frac{E_j}{k_B T}\right)}$$

where k_B is the Boltzmann constant and T is the thermodynamic temperature.
For velocity $\mathbf{v}$ and $E = \frac{1}{2}m\mathbf{v}^2$, and calculate its normalizing constant in the momentum space, we obtain

Maxwell-Boltzmann Distribution for Velocity

$$f_{\mathbf{v}}(v_x, v_y, v_z) = \left[\frac{m}{2\pi k_B T}\right]^{3/2} \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2k_B T}\right)$$

Basics: Optical tweezers – Ray optics

Conservation law of momentum for particles and photons

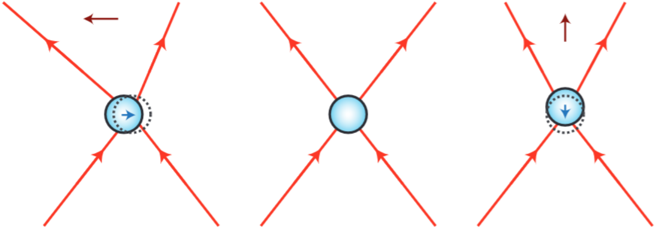


Figure 2.1: Qualitative view of optical trapping of dielectric spheres. (A) displays the force on the particle when the particle is displaced laterally from the focus; (B) shows that there is no net force on the particle when the particle is trapped at the focus; and (C) displays the force on the particle when the particle is positioned above the focus.

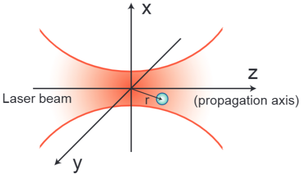
Basics: Optical tweezers – Gaussian beam

For particles trapped in a Gaussian beam, it experiences a potential:

$$V(\mathbf{r}) = -\frac{2\pi n_m R^3}{c}\left(\frac{m^2-1}{m^2+2}\right) I(\mathbf{r})$$

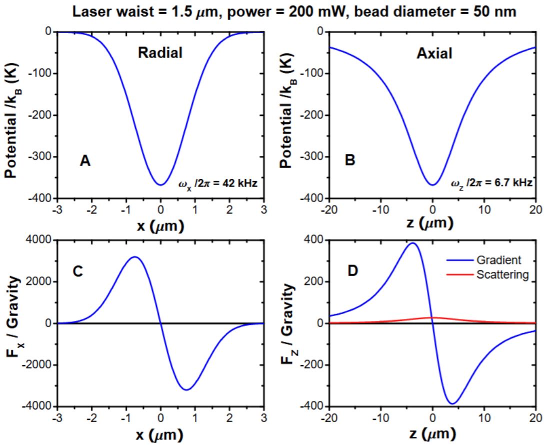
where n_m is the refractive index of the medium, R represents the particle radius, and $m = n_p/n_m$ is the relative refractive index between the particle and medium.

$I(\mathbf{r}) = \frac{2P}{\pi\omega(z)^2} e^{-\frac{2(x^2+y^2)}{\omega(z)^2}}$, $\omega(z)$ is the beam waist, P is the total power of the beam.



Basics: Optical tweezers – Gaussian beam

For small displacements, the effect of optical tweezers on the bead's motion can be approximated by a harmonic potential.



Basics: Brownian motion – Free particle

Langevin equation (1D)[1]:

$$m\frac{dv}{dt} = -\frac{v}{B} + F(t); \quad \langle F(t) \rangle = 0.$$

$$\langle v(t) \rangle = v(0)\exp(-t/\tau), \quad (\tau = mB).$$

where B is the mobility of the system[2], and τ is the relaxation time.
Multiply both sides by x and take the ensemble average:

$$\frac{d^2}{dt^2}\langle x^2 \rangle + \frac{1}{\tau}\frac{d}{dt}\langle x^2 \rangle = 2\langle v^2 \rangle.$$

If the Brownian particle has already attained thermal equilibrium, i.e. $\langle v^2 \rangle = k_B T/m$ by the energy equipartition theorem. We have the result:

$$\langle r^2 \rangle = \frac{2k_B T\tau^2}{m}\left\{\frac{t}{\tau} - \left(1 - e^{-t/\tau}\right)\right\}.$$

[1] R.K. Pathria, P.D. Beale, *Statistical Mechanics* (3rd Ed.), Academic Press, 2011, pp. 583-635.

[2] If Stokes's law is applicable, then $B = 1/6\pi\eta a$, where η is the coefficient of viscosity of the fluid and a the radius of the particle.

Basics: Brownian motion – Free particle

When $t \ll \tau$, it's called the **ballistic** regime:

$$\langle x^2 \rangle \simeq \frac{k_B T}{m} t^2 = \langle v^2 \rangle t^2$$

The dynamics of a particle is dominated by its inertia with *reversible* nature of motion.
On the other hand, when $t \gg \tau$, it's called the **diffusive** regime with its *irreversible* nature:

$$\langle x^2 \rangle \simeq \frac{2 k_B T \tau}{m} t = (2 B k_B T) t.$$

We obtain:

Einstein Relation

$$D = B k_B T$$

Basics: Brownian motion – Harmonic oscillator

The optical tweezers provide a harmonic oscillator potential with spring constant $m\omega_0^2$ for small displacements. The Langevin equation is given by:

$$\frac{d^2x}{dt^2} + \frac{1}{B}\frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m},$$

And its solutions are modified by the oscillatory terms:

Mean square displacement (MSD): $\left\langle [\Delta x(t)]^2 \right\rangle = \frac{2k_B T}{m\omega_0^2}\left[1 - e^{-t/2\tau}\left(\cos\omega_1 t + \frac{\sin\omega_1 t}{2\omega_1\tau}\right)\right]$

Normalized velocity autocorrelation function (VACF): $\psi(t) = e^{-t/2\tau}\left(\cos\omega_1 t - \frac{\sin\omega_1 t}{2\omega_1\tau}\right)$

where $\omega_1 = \sqrt{\omega_0^2 - 1/(2\tau)^2}$.
The asymptotic behaviors of the solutions remain the same for ballistic regime.

Select experimental subjects

3 μm **silica beads trapped in air.**

Medium: Air or Water?

The viscosity is much lower in the air with a relaxation time $\tau \approx 100\ \mu s$, compared to 0.1 μs in water. The root mean square velocity is given by $v_{rms} \approx 0.5\ mm/s$.

Trapping in air can reduce the spatial resolution requirements to 0.5 nm in 10 μs (with 10% uncertainty).

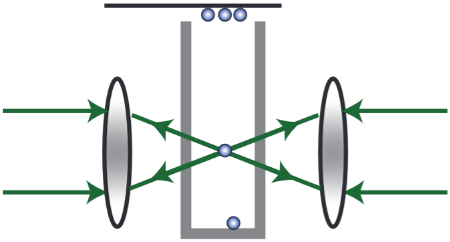
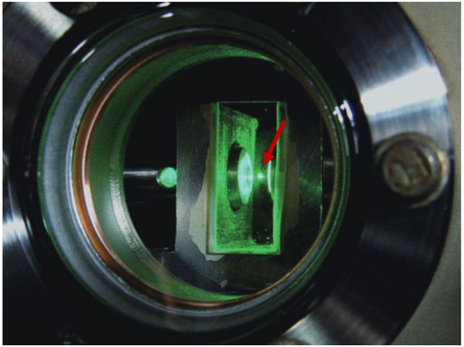
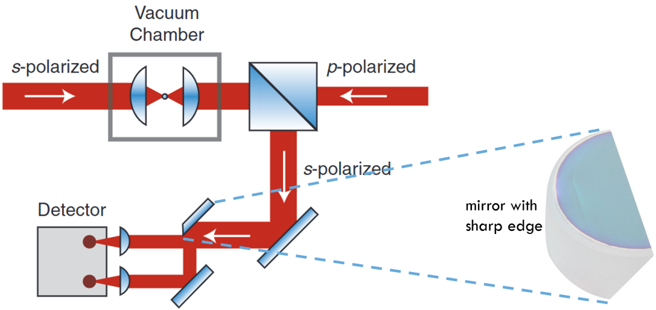


Figure 3.9: A counter-propagating dual-beam optical trap in air.



Schematic for the setup

Two counter-propagating orthogonally polarized light beams(frequency differed by 160 MHz to prevent interference) create a three-dimensional harmonic potential trap for the bead.
The s-polarized light is reflected by PBS, further split by a mirror with sharp edge[3] and measured by a 75 MHz balanced detector. Projection of the Brownian motion in one direction.
The position is determined by $x = \alpha(V_1 - V_2)$. α is the voltage-to-position calibration factor.



[3] https://www.thorlabs.com/thorproduct.cfm?partnumber=BBD05-E03

Results: Trajectories in two air pressures

Higher pressure → Higher viscosity (shorter relaxation time) → Increased velocity randomization

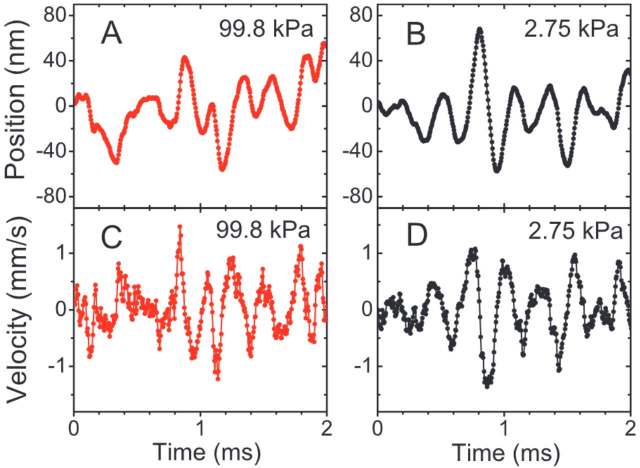
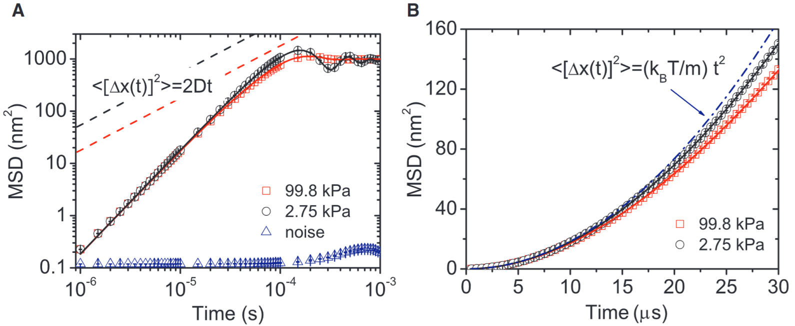


Fig. 2. One-dimensional trajectories of a 3-µm-diameter silica bead trapped in air at 99.8 kPa (**A**) and

Results: Mean square displacement

$$\left\langle [\Delta x(t)]^2 \right\rangle = \frac{2k_BT}{m\omega_0^2}\left[1 - e^{-t/2\tau}\left(\cos\omega_1 t + \frac{\sin\omega_1 t}{2\omega_1\tau}\right)\right], \quad \langle \Delta x^2 \rangle \simeq \frac{k_BT}{m}t^2 \quad t \ll \tau$$

The "noise" signal is recorded when there is no particle in the optical trap.

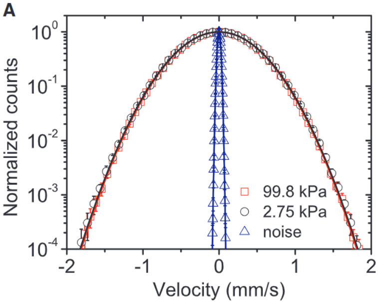


Results: Maxwell-Boltzmann velocity distribution

$$f_{MB}(v) = \left[\frac{m}{2\pi k_B T}\right]^{1/2} \exp\left(-\frac{mv^2}{2k_B T}\right), \quad v_{rms,\ theory} = \sqrt{\frac{k_B T}{m}} = 0.429\ \text{mm/s}$$

$$v_{rms,\ 99.8\ \text{kPa}} = 0.422\ \text{mm/s}, \quad v_{rms,\ 2.75\ \text{kPa}} = 0.425\ \text{mm/s}$$

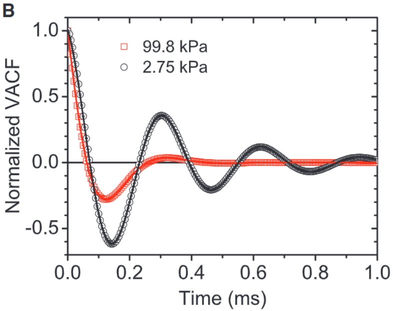
Both are as expected (within $v_{rms,\ noise} = 0.021$ mm/s) and independent of pressure.



Results: Normalized velocity autocorrelation function

$$\psi(t) = e^{-t/2\tau} \left(\cos \omega_1 t - \frac{\sin \omega_1 t}{2\omega_1 \tau} \right)$$

99.8 kPa: $\tau = 48.5 \pm 0.1$ ms, $\omega_0 = 2\pi \cdot (3064 \pm 4)$ Hz.
2.75 kPa: $\tau = 147.3 \pm 0.1$ ms, $\omega_0 = 2\pi \cdot (3168 \pm 0.5)$ Hz.



Conclusion and outlook

- Successfully measured the instantaneous velocity of a Brownian particle trapped in air using optical tweezers.
- Direct verification of the Maxwell-Boltzmann velocity distribution and the equipartition theorem.
- Observation of the ballistic regime of Brownian motion at short timescales.
- Potential applications in fundamental tests of statistical mechanics and cooling particles to the quantum ground state.

Thank you for your attention!