Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

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The paper[1]

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

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Josephson Junction

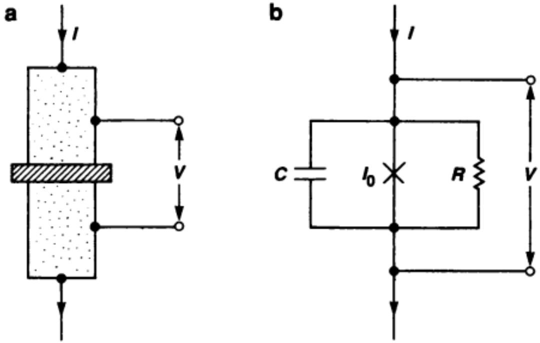


Figure: Schematic **(a)** and circuit **(b)** representation of Josephson Junction

Josephson current-phase relation

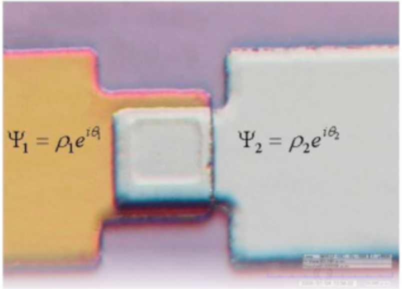


Figure: Two bulk superconducting electrodes can be viewed phenomenologically as possessing separate macroscopic wavefunctions Ψ_1 and Ψ_2, with phase difference $\delta = \theta_1 - \theta_2$.[1]

Josephson current-phase relation

$$I = I_0 \sin \delta$$

[1] J. A. Blackburn, Matteo Cirillo, and Niels Grønbech-Jensen. "A survey of classical and quantum interpretations of experiments on Josephson junctions at very low temperatures". In: Physics Reports 611 (2016), pp. 1–33. ISSN: 0370-1573. DOI: https://doi.org/10.1016/j.physrep.2015.10.010

Josephson Voltage-Frequency relation

When I exceeds I_0, a voltage is developed across the junction and δ evolves with time according to

Josephson Voltage-Frequency relation

$$\dot{\delta} = 2\pi V/\Phi_0,$$

where $\Phi_0 = h/2e$ is the flux quantum.

Josephson Junction Dynamics (1/3)

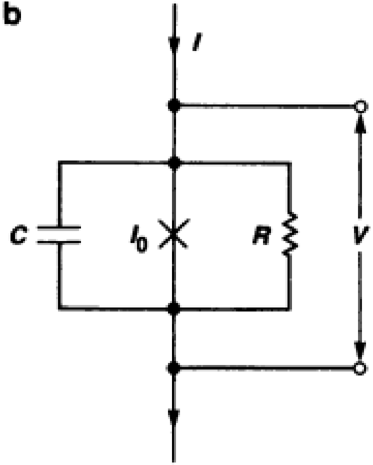
Classical equation of motion for the phase difference

$$C\left(\frac{\Phi_0}{2\pi}\right)^2\ddot{\delta}+\frac{1}{R}\left(\frac{\Phi_0}{2\pi}\right)^2\dot{\delta}+\frac{\partial U(\delta)}{\partial\delta}=\frac{\Phi_0}{2\pi}I_N(t) \quad (1)$$

The term $I_N(t)$ represents the Nyquist current noise generated by the resistor R at temperature T, and

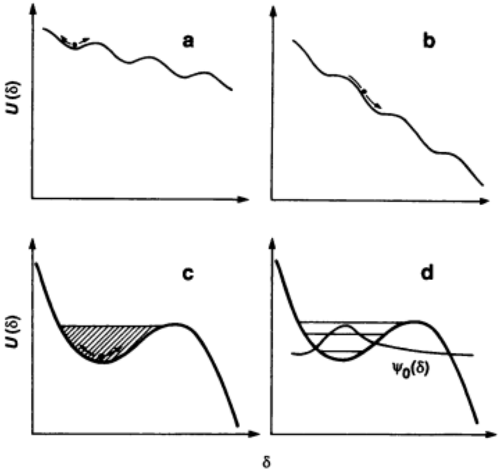
$$U(\delta)=-\left(\frac{I_0\Phi_0}{2\pi}\right)\left[\cos\delta+\left(\frac{I}{I_0}\right)\delta\right]$$

Classical equation of motion for coordinate δ, and mass $C\left(\frac{\Phi_0}{2\pi}\right)^2$ moving in the tilted washboard potential $U(\delta)$.



Josephson Junction Dynamics (2/3)

Fig. 2. Tilted washboard analog of Josephson tunnel junction: (**a**) stationary state ($V = 0$) for $I < I_0$, and (**b**) running state ($V \neq 0$) for $I > I_0$. In the stationary state in the classical regime (**c**) the particle is point-like with a continuous energy range, whereas in (**d**) the ground state $\psi_0(\delta)$ of the particle is described by a wave packet and the energy is quantized into levels.



Josephson Junction Dynamics (3/3)

The particle[2] oscillates at the bottom of the well at the so-called plasma frequency

$$\omega_p = \left(\frac{2\pi I_0}{\Phi_0 C}\right)^{1/2} \left[1 - \left(\frac{I}{I_0}\right)^2\right]^{1/4}$$

In the experiments to be described, I is very close to I_0, and the potential is of the form $A\delta^2 - B\delta^3$ $(A, B > 0)$. In this approximation, the barrier height is

$$\Delta U = \left[\frac{2^{1/2} I_0 \Phi_0}{3\pi}\right] \left(1 - \frac{I}{I_0}\right)^{3/2}$$

The damping of the oscillations by the resistance R (assumed to be linear) is represented by

$$Q = \omega_p R C$$

[2] The exact correspondence between the motion of the particle and the dynamics of δ is very useful, since it provides a heuristic model with which one can understand the dynamics of the junction.

Kramers' escape rate

In this classical description, the particle can escape from the well as a result of thermal activation. The escape rate for thermal activation is given by :

Kramers' escape rate

$$\Gamma_t(T) = a_t \left(\frac{\omega_p}{2\pi}\right) \exp\left(-\frac{\Delta U}{k_B T}\right)$$

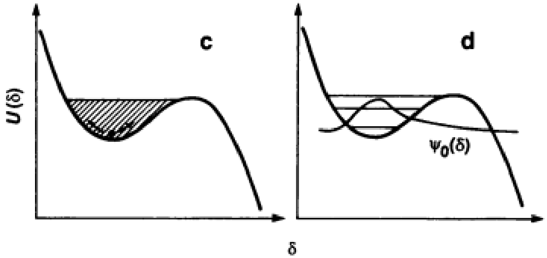
where the prefactor a_t is of order unity. The thermal energy of the particle arises from the noise current $I_N(t)$. In thermal activation the system is entirely classical and is described by a classical equation of motion with a continuous energy range.

From Classical to Quantum

Critical temperature

$$T_{\text{cr}} = \frac{\hbar\omega_p}{2\pi k_B} \quad (\text{for } Q \gg 1)$$

Below this temperature the position of the particle is described by a wave packet, $\psi(\delta)$, and the energy of the well by **Macroscopic quantum tunneling (MQT)** through the barrier.



Macroscopic Quantum Tunneling

MQT Escape rate to first order in $1/Q$ at $T = 0$

$$\Gamma_q(0) = \left[120\pi\left(\frac{7.2\Delta U}{\hbar\omega_p}\right)\right]^{1/2}\frac{\omega_p}{2\pi}\exp\left[-7.2\frac{\Delta U}{\hbar\omega_p}\left(1+\frac{0.87}{Q}\right)\right] \tag{2}$$

- The reduction of $\Gamma_q(0)$ by dissipation arises from a narrowing of the wave packet.
- In the limit $Q \to \infty$, $\Gamma_q(0)$ reduces to the Wentzel-Kramer-Brillouin (WKB) result.

Distinction between Josephson tunneling and macroscopic quantum tunneling

In the process of macroscopic quantum tunneling, it is the particle associated with the phase difference δ that tunnels, as opposed to the tunneling of individual Cooper pairs that occurs in Josephson tunneling.

Determination of Junction Parameters in the Classical Regime (1/2)

- Parameters ω_p and Q using resonant activation.
- Measurement of the parameters I_0, C, and R in the classical regime.
- $\ln\left(\omega_p/I\right)/2\pi\Gamma(I))^{2/3}$ versus I should yield a straight line with slope scaling as $T^{-2/3}$ that intersects the current axis very close to I_0.

Barrier height and Escape rate

$$\Delta U = \left[\frac{2^{1/2} I_0 \Phi_0}{3\pi}\right]\left(1 - \frac{I}{I_0}\right)^{3/2} \quad \text{and} \quad \Gamma_t(T) = a_t\left(\frac{\omega_p}{2\pi}\right)\exp\left(-\frac{\Delta U}{k_B T}\right)$$

Damping rate and Plasma frequency

$$Q = \omega_p RC \quad \text{and} \quad \omega_p = \left(\frac{2\pi I_0}{\Phi_0 C}\right)^{1/2}\left[1 - \left(\frac{I}{I_0}\right)^2\right]^{1/4}$$

Determination of Junction Parameters in the Classical Regime (2/2)

Table 1. Measured parameters for a shunted and unshunted Josephson tunnel junction, with experimental (T^{e}_{esc}) and predicted (T^{p}_{esc}) escape temperatures at $T = 0$ extrapolated from results at higher temperatures. The predicted values of T^{p}_{esc} for $Q = \infty$ are also included for comparison.

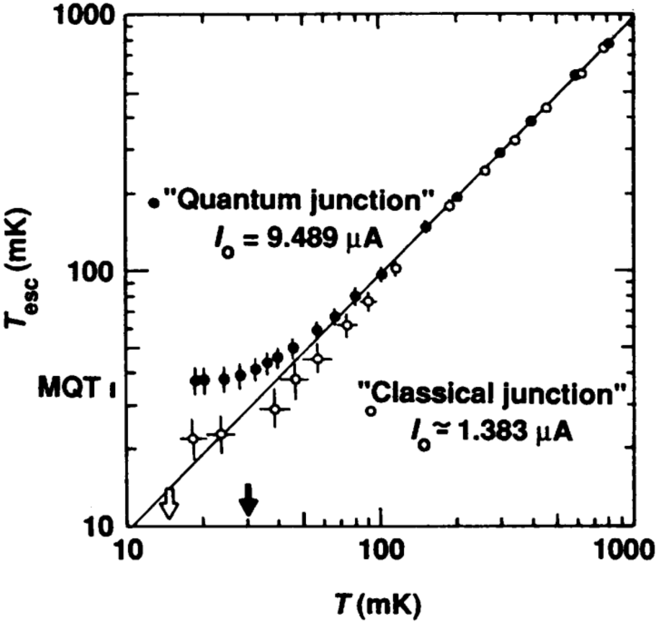
Quantity	Unshunted junction	Shunted junction
I_o (μA)	9.489 ± 0.007	24.873 ± 0.004
C (pF)	6.35 ± 0.4	4.28 ± 0.34
R (ohms)	190 ± 100	9.3 ± 0.1
Q	30 ± 15	1.77 ± 0.07
T^{e}_{esc} (mK)	37.4 ± 4.0	44.4 ± 1.7
T^{p}_{esc} (mK)	36.0 ± 1.4	42.5 ± 2.1
T^{p}_{esc} ($Q = \infty$)	37.5 ± 1.4	69 ± 3

Macroscopic Quantum Tunneling (1/2)

It is convenient to express our escape rates in both classical and quantum regimes in terms of an escape temperature T_{esc} defined through the relation

$$\Gamma = \left(\frac{\omega_p}{2\pi}\right) \exp\left(-\frac{\Delta U}{k_B T_{\text{esc}}}\right) \tag{3}$$

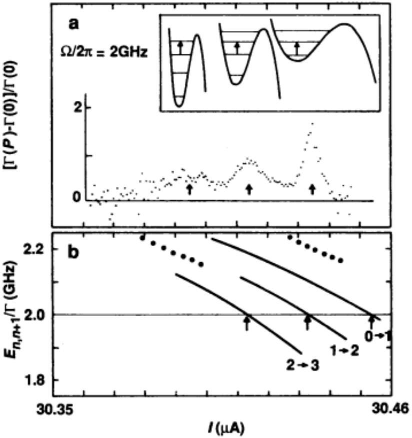
In the classical regime, T_{esc} is very nearly equal to T with a small correction due to the departure of a_t from unity. In the quantum regime, T_{esc} takes a temperature-independent value that can be calculated exactly by comparing Eqs. (2) and (3).



1000
100
10
10
100
1000
T_{esc} (mK)
T (mK)
MQT
• "Quantum junction"
I_o = 9.489 μA
"Classical junction"
o
I_o ≃ 1.383 μA

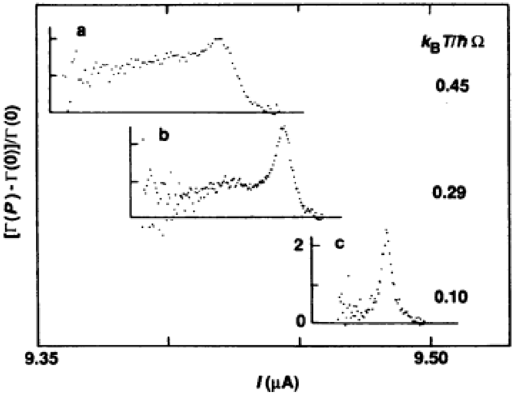
Quantized Energy Levels (1/2)

Fig. 6. (**a**) $[\Gamma(P) - \Gamma(0)]/\Gamma(0)$ versus I for an 80 by 10 μm^2 junction at 28 mK in the presence of 2.0 GHz microwaves ($k_B T/\hbar\Omega = 0.29$). Arrows indicate values of current at which the peaks of the resonances occur. Inset represents the corresponding transitions between energy levels. (**b**) Calculated energy-level spacings $E_{n,n+1}$ versus I for $I_0 = 30.572 \pm 0.017$ μA and $C = 47.0 \pm 3.0$ pF. Dotted lines indicate uncertainties in the $0 \rightarrow 1$ curve that arise from uncertainties in I_0 and C. Arrows indicate values of bias current at which resonances are predicted to occur.



Quantized Energy Levels (2/2)

Fig. 7. $[\Gamma(P) - \Gamma(0)]/\Gamma(0)$ versus I for a 10 by 10 μm^2 junction with $I_0 \approx 9.57$ μA and $C \approx 6.35$ pF at three values of $k_B T/\hbar\Omega$. The microwave frequencies are: curve a, 4.5 GHz; curve b, 4.1 GHz; and curve c, 3.7 GHz.



Questions?

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[1] John Clarke et al. “Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction”. In: Science 239.4843 (1988), pp. 992–997. DOI: 10.1126/science.239.4843.992. URL: https://doi.org/10.1126/science.239.4843.992.

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