Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.4

Solution: The Generalized Fluctuation Dissipation Theorem

1. Start from the equation of motion for a harmonic oscillator with mass m, damping γ and resonance frequency ω_0 in time and frequency domain

$$m\frac{d^2}{dt^2}x + \gamma\frac{d}{dt}x + \omega_0^2 x = F_L(t), \tag{1}$$

$$x(\omega) = \frac{F_L(\omega)}{m(\omega_0^2 - \omega^2) - i\gamma\omega}. \tag{2}$$

Here the adopted Fourier transform notation is $f(t) = \int f(\omega)e^{-i\omega t}d\omega$. Mechanical impedance and resistance:

$$Z(\omega) = \frac{F_L(\omega)}{v(\omega)} = \frac{F_L(\omega)}{-i\omega x(\omega)} = \gamma + im\frac{\omega_0^2 - \omega^2}{\omega}, \tag{3}$$

$$R(\omega) = \mathrm{Re}\, Z(\omega) = \gamma. \tag{4}$$

2. Spectrum of position fluctuations:

$$S_{xx}(\omega) = \frac{1}{m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2} S_{F_L F_L}(\omega) = \frac{4kT\gamma}{m^2(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}. \tag{5}$$

3. Generic equation of motion for an oscillator in frequency domain:

$$x(\omega) = \frac{F_L(\omega)}{(k - m\omega^2) - ik\phi(\omega)}, \tag{6}$$

where $\phi(\omega)$ is the loss angle.

$$R(\omega) = k\phi(\omega)/\omega, \tag{7}$$

$$S_{xx}(\omega) = \frac{4kTk\phi(\omega)}{\omega((k - m\omega^2)^2 + (k\phi(\omega))^2)}, \tag{8}$$

$$S_{vv}(\omega) = \omega^2 S_{xx}(\omega). \tag{9}$$

Solution: Stationary solutions of the Fokker Planck Equation

The 1 dimensional Fokker Planck equation with constant drift (Smoluchowski equation) is given by: $\frac{\partial}{\partial t}P(x,t) = -\frac{1}{\gamma}\frac{\partial}{\partial x}(F(x)P(x,t)) + D \cdot \frac{\partial^2}{\partial x^2}P(x,t)$, where D is a the diffusion constant, $F(x)$ the conservative force and γ is the dissipation constant. This equation can also be written in the form of a a continuity equation:

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}J(x,t)$$

Where the probability current is $J(x,t) = \left[\frac{1}{\gamma}F(x) - D \cdot \frac{\partial}{\partial x}\right] P(x,t)$.

1. Having $J = 0$,

$$D \cdot \frac{d}{dx}\ln(P(x)) = \frac{1}{\gamma}F(x),$$

$$P(x) = Ne^{-\Phi(x)},\ \Phi(x) = \frac{1}{D\gamma}\int_0^x F(x')dx' = -\frac{V(x)}{D\gamma} = -\frac{V(x)}{k_B T}$$

Above N is a normalization constant.

2. Integrating the continuity equation we get $\frac{\partial}{\partial t}\int_{x_{min}}^{x_{max}} P(x,t)dx = J(x_{min}) - J(x_{max}) = 0$ so $\int_{x_{min}}^{x_{max}} P(x,t)dx = const$.

3. Look for steady state solution, so $J(x_{min}) = J(x_{max}) = J$ implies $J(x) = J$. We then have an equation
$$J = \frac{1}{\gamma}F(x)P(x) - D \cdot \frac{d}{dx}P(x),$$
which can be solved as
$$P(x) = C(x)e^{-\frac{V(x)}{\gamma D}},\ D\frac{dC}{dx} = -Je^{\frac{V(x)}{\gamma D}},$$
$$P(x) = P(0)e^{-\Phi(x)} - \frac{J}{D}\int_0^x e^{-(\Phi(x)-\Phi(x'))}dx'.$$

Solution: Fokker Planck Equation to derive the limit of atomic laser cooling

1. For the random walk, we can write down the update formula
$$P(p,t+\Delta t) - P(p,t) = P(p+\Delta p,t)\epsilon_-(p+\Delta p)\Delta t + P(p-\Delta p,t)\epsilon_+(p-\Delta p)\Delta t - \\ - P(p,t)[\epsilon_-(p)+\epsilon_+(p]\Delta t$$
where $\Delta p = \hbar k$. We can expand the first two terms on the RHS up to second order as
$$P(p\pm\Delta p,t)\epsilon_\mp(p\pm\Delta p) \approx P(p,t)\epsilon_\mp(p) \pm \Delta p\frac{\partial}{\partial p}\left[P(p,t)\epsilon_\mp(p)\right] + \\ + \frac{\Delta p^2}{2}\frac{\partial^2}{\partial p^2}\left[P(p,t)\epsilon_\mp(p)\right] \quad (10)$$
so the equation becomes
$$\frac{P(p,t+\Delta t) - P(p,t)}{\Delta t} \approx \Delta p\frac{\partial}{\partial p}\left[P(p,t)(\epsilon_-(p)-\epsilon_+(p))\right] + \\ + \frac{\Delta p^2}{2}\frac{\partial^2}{\partial p^2}\left[P(p,t)(\epsilon_+(p)+\epsilon_-(p))\right] \quad (11)$$
Taking the limit and using the definitions for $\epsilon_\pm$we obtain
$$\partial_t P(p,t) = \partial_p[\beta v P(p,t)] + \partial_p^2\left[D(p,t)P(p,t)\right] \quad (12)$$
This is a Fokker-Planck eq. with $A(p,t) = -\beta\frac{p}{m} = -\beta v$ and $\beta = sk^2\hbar/(\gamma/2)$, $D = sk^2\hbar^2$.

2. We are looking at the steady state
$$0 = \partial_p[\beta\frac{p}{m}P(p,t)] + \partial_p^2\left[D \cdot P(p,t)\right] \quad (13)$$
After one integration we get
$$\frac{\partial_p P(p,t)}{P(p,t)} = -\frac{\beta p}{Dm} \quad (14)$$

3. In a general case the solution of Equation 13 is found as
$$P(p) = \frac{C}{D(p)}\exp\left(-\int_0^p \frac{\beta(p')p'}{mD(p')}dp'\right), \quad (15)$$
where C is a normalization constant.

Given the substitution

$$\beta \to \frac{\beta}{1+(v/v_c)^4}, \tag{16}$$

$$D \to D\frac{1+(v/v_c)^2}{1+(v/v_c)^4}, \tag{17}$$

we find

$$P(p) = \frac{C}{D}\frac{1+(p/p_c)^4}{1+(p/p_c)^2}(1+(p/p_c)^2)^{-\frac{\beta p_c^2}{2mD}}, \tag{18}$$

where $p_c = mv_c$.