Statistical Physics IV: Non-equilibrium statistical physics
ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to Exercise No.10

Solution: Standard quantum limit for gravitational waves detection

The problem follows loosely the second problem of the K-lab homework Quantum limited Displacement Measurements (see detailed attached solution).

1. Generic Hamiltonian of the system:

$$\hat{H} = \hbar\left(\omega_0 + \sqrt{2}g_0\hat{x}\right)\hat{a}^\dagger\hat{a} + \hbar\Omega_\mathrm{m}\left(\frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}\right) - F_{GW}(t)\hat{x} + \hat{H}_{sys-bath}, \tag{1}$$

where $\hat{a}$ is the light field operator, Ω_m is the oscillator frequency and $\hat{H}_{sys-bath}$ describes coupling of the mechanical and optical fields to the thermal bath modes.

2. Linearization around large coherent amplitudes

$$\hat{a} = \alpha + \delta\hat{a},$$
$$\hat{x} = \bar{x} + \delta\hat{x},$$
$$\hat{p} = \delta\hat{p},$$

up to the quadratic terms in $\delta\hat{a}, \delta\hat{x}$

$$\hat{H} = \hbar(\omega_c - \omega_L + \sqrt{2}g_0\bar{x})(\alpha^2 + \alpha(\delta\hat{a} + \delta\hat{a}^\dagger) + \delta\hat{a}^\dagger\delta\hat{a}) + \hbar\sqrt{2}g_0\alpha(\delta\hat{a} + \delta\hat{a}^\dagger)\delta\hat{x} + \\ + \hbar\Omega_\mathrm{m}\left(\frac{\delta\hat{p}^2}{2} + \frac{\delta\hat{x}^2}{2} + \bar{x}\delta\hat{x} + \frac{\bar{x}^2}{2}\right) - F_{GW}(t)\delta\hat{x} - F_{GW}(t)\bar{x} + \hat{H}_{sys-bath}. \tag{2}$$

Note that the cavity energy term vanishes in the frame, rotating for the optical field at the shifted cavity frequency $\omega_0 + \sqrt{2}g_0\bar{x}$. We can find $\bar{x}$ by neglecting fluctuations

$$\frac{\partial H}{\partial \bar{x}} = 0,$$
$$\bar{x} = \frac{\sqrt{2}g_0}{\Omega_m}\alpha^2.$$

3. The quantum Langevin equations (in the high-Q approximation $\kappa \ll \omega_0$, $\Gamma_\mathrm{m} \ll \Omega_m$):

$$\dot{\delta\hat{a}} = -\frac{\kappa}{2}\delta\hat{a} - ig(\delta\hat{b} + \delta\hat{b}^\dagger) + \sqrt{\kappa}\,\delta\hat{a}_\mathrm{in} \tag{3}$$
$$\dot{\delta\hat{b}} = -i\Omega_\mathrm{m}\delta\hat{b} - \frac{\Gamma_\mathrm{m}}{2}\delta\hat{b} - ig(\delta\hat{a} + \delta\hat{a}^\dagger) + \frac{i}{\sqrt{2}}ks(t), \tag{4}$$

where $g = g_0\alpha$ is the multi-photon optomechanical cooperativity and $F_{GW}(t) = \hbar ks(t)$. In terms of the quadratures:

$$\dot{\delta\hat{Y}} = -\frac{\kappa}{2}\delta\hat{Y} - 2g\delta\hat{x} + \sqrt{\kappa}\,\delta\hat{Y}_\mathrm{in}, \tag{5}$$
$$\dot{\delta\hat{X}} = -\frac{\kappa}{2}\delta\hat{X} + \sqrt{\kappa}\,\delta\hat{X}_\mathrm{in}, \tag{6}$$
$$\dot{\delta\hat{x}} = \Omega_\mathrm{m}\delta\hat{p} - \frac{\Gamma_\mathrm{m}}{2}\delta\hat{x}, \tag{7}$$
$$\dot{\delta\hat{p}} = -\Omega_\mathrm{m}\delta\hat{x} - \frac{\Gamma_\mathrm{m}}{2}\delta\hat{p} - 2g\delta\hat{X} + ks(t). \tag{8}$$

4. Modulation of the intracavity field phase by mechanical motion:

$$\delta\hat{Y}[\omega] = \frac{1}{-i\omega+\kappa/2}(-2g\,\delta\hat{x}[\omega] + \sqrt{\kappa}\delta\hat{Y}_{\text{in}}[\omega]), \tag{9}$$

5. Backaction- and signal- driven motion of the mechanical oscillator

$$\begin{aligned}\delta\hat{x}[\omega] = \frac{\Omega_{\text{m}}}{(i\omega-\Gamma_{\text{m}}/2)^2+\Omega_{\text{m}}^2}\left(-2g\delta\hat{X} + \sqrt{2}ks[\omega]\right) = \\ = \frac{\Omega_{\text{m}}}{(i\omega-\Gamma_{\text{m}}/2)^2+\Omega_{\text{m}}^2}\left(-\frac{2g\sqrt{\kappa}}{-i\omega+\kappa/2}\delta\hat{X}_{\text{in}} + ks[\omega]\right).\end{aligned} \tag{10}$$

Here the oscillator susceptibility to force, $\chi_{\text{m}}[\omega] = \Omega_{\text{m}}/(\Omega_{\text{m}}^2-\omega^2-i\Gamma_{\text{m}}\Omega) \approx \Omega_{\text{m}}/((i\omega-\Gamma_{\text{m}}/2)^2+\Omega_{\text{m}}^2)$, looks unusual due to the adopted high-Q approximation.

6. Total output signal on phase quadrature

$$\begin{aligned}\delta\hat{Y}_{\text{out}}[\omega] = \delta\hat{Y}_{\text{in}}[\omega] - \sqrt{\kappa}\delta\hat{Y}[\omega] = \frac{i\omega+\kappa/2}{i\omega-\kappa/2}\delta\hat{Y}_{\text{in}}[\omega] + \frac{2g\sqrt{\kappa}}{-i\omega+\kappa/2}\delta\hat{x} = \\ = \frac{i\omega+\kappa/2}{i\omega-\kappa/2}\delta\hat{Y}_{\text{in}}[\omega] - \frac{4g^2\kappa}{(-i\omega+\kappa/2)^2}\chi_{\text{m}}[\omega]\delta\hat{X}_{\text{in}} + \frac{2g\sqrt{\kappa}}{-i\omega+\kappa/2}\chi_{\text{m}}[\omega]ks[\omega] = \\ = A[\omega]\delta\hat{Y}_{\text{in}}[\omega] + B[\omega]\delta\hat{X}_{\text{in}} + C[\omega]s[\omega].\end{aligned} \tag{11}$$

7. Correlators of the quadratures of the field:

$$\langle\delta\hat{X}_{\text{in}}(t)\delta\hat{X}_{\text{in}}(t')\rangle = \langle\delta\hat{Y}_{\text{in}}(t)\delta\hat{Y}_{\text{in}}(t')\rangle = \frac{1}{2}\delta(t-t'), \tag{12}$$

$$\langle\delta\hat{X}_{\text{in}}(t)\delta\hat{X}_{\text{in}}(t')\rangle = \frac{i}{2}\delta(t-t'), \tag{13}$$

$$\left\langle\left(\delta\hat{X}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{X}_{\text{in}}[\omega]\right\rangle = \left\langle\left(\delta\hat{Y}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{Y}_{\text{in}}[\omega]\right\rangle = \frac{1}{2}, \tag{14}$$

$$\left\langle\left(\delta\hat{X}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{Y}_{\text{in}}[\omega]\right\rangle = \frac{i}{2}, \tag{15}$$

Spectrum of the output signal, introducing the cooperativity parameter $C = 4g^2/\kappa\Gamma_{\text{m}}$:

$$\begin{aligned}\left\langle\left(\delta\hat{Y}_{\text{in}}[\omega]\right)^{\dagger}\delta\hat{Y}_{\text{in}}[\omega]\right\rangle_s = \frac{1}{2} + \frac{8C^2\Gamma_{\text{m}}^2}{(1+(2\omega/\kappa)^2)^2}|\chi_{\text{m}}[\omega]|^2 + \\ + \frac{4C\Gamma_{\text{m}}}{1+(2\omega/\kappa)^2}|\chi_{\text{m}}[\omega]|^2k^2\left\langle(s[\omega])^{\dagger}s[\omega]\right\rangle\end{aligned} \tag{16}$$

8. —

9. In the Eq. 16 the 3-rd term is signal nad the remaining are noises. Minimization over α is equivalent to minimization over C, so we obtain (if $\omega \ll \kappa/2$)

$$C_{\text{SQL}} = \frac{1}{4\Gamma_{\text{m}}|\chi_{\text{m}}[\omega]|}, \tag{17}$$

$$g_{\text{SQL}} = \frac{1}{4}\sqrt{\frac{\kappa}{|\chi_{\text{m}}[\omega]|}}, \tag{18}$$

$$(\text{Signal/Noise})_{\text{SQL}} = k^2\left\langle(s[\omega])^{\dagger}s[\omega]\right\rangle \times |\chi_{\text{m}}[\omega]|, \tag{19}$$

$$S_{\text{noise,SQL}}[\omega] = \frac{1}{k^2|\chi_m[\omega]|}. \tag{20}$$

Solution: Quantum regression theorem and photon bunching[1]

We use the quantum regression theorem

$$\left\langle \hat{O}_1(t)\hat{A}_\mu(t+\tau)\hat{O}_2(t)\right\rangle = \sum_\nu M_{\mu\nu}\left\langle \hat{O}_1(t)\hat{A}_\nu(t+\tau)\hat{O}_2(t)\right\rangle ,$$

with operators $A_1 = a^\dagger a$, $A_2 = 1$. We have

$$\begin{pmatrix} \langle \dot{A}_1\rangle \\ \langle \dot{A}_2\rangle \end{pmatrix} = \begin{pmatrix} -\gamma & \gamma\bar{n} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \langle A_1\rangle \\ \langle A_2\rangle \end{pmatrix} \tag{21}$$

With $O_1(t) = a^\dagger(t)$ and $O_2(t) = a(t)$, we get

$$\frac{d}{d\tau}\left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\right\rangle = -\gamma\left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\right\rangle + \gamma\bar{n}\left\langle a^\dagger(t)a(t)\right\rangle \tag{22}$$

We can solve it as

$$\left\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\right\rangle = \langle m\rangle e^{-\gamma\tau} + \bar{n}\langle n\rangle (1-e^{-\gamma\tau}) \tag{23}$$

with $\langle m\rangle = \langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle$.

We still need to derive the evolution of $\langle a^\dagger(t)a^\dagger(t)a(t)a(t)\rangle$ using the master equation.

$$\begin{aligned}\langle \dot{m}\rangle =& Tr[m\dot{\rho}]\\ =& -i\omega_0 Tr[a^{\dagger 2}a^2a^\dagger a\rho - a^{\dagger 2}a^2\rho a^\dagger a] + \frac{\gamma}{2}(2a^{\dagger 2}a^3\rho a^\dagger - a^{\dagger 2}a^2a^\dagger a\rho - a^{\dagger 2}a^2\rho a^\dagger a)\\ &+\gamma\bar{n}(a^{\dagger 2}a^2a\rho a^\dagger + a^{\dagger 2}a^2a^\dagger\rho a - a^{\dagger 2}a^2a^\dagger a\rho - a^{\dagger 2}a^2\rho a a^\dagger)\end{aligned} \tag{24}$$

Using $[a, a^\dagger] = 1$ and $Tr[ABC] = Tr[CAB] = Tr[BCA]$, we find the first term is 0, second term equals to $-2\gamma\langle a^{\dagger 2}a^2\rangle$ and the third term is $4\gamma\bar{n}\langle n\rangle$.

So we gets

$$\langle \dot{m}\rangle = -2\gamma\langle m\rangle + 4\gamma\bar{n}\langle n(t)\rangle \tag{25}$$

Now the shortcut to the answer is to assume that in the long time t limit, the solution is stationary, so $\langle \dot{m}\rangle = 0$. This means $\langle n\rangle = \bar{n}$ and $\langle m\rangle = 2\bar{n}^2$.

Finally, putting everything together and neglecting the terms in $e^{-\gamma t}$ in the long time t limit, one obtains

$$\left\langle \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t)\right\rangle = \bar{n}^2\left(1+e^{-\gamma\tau}\right).$$

Solution: Asymmetry of the spectral density of the quantum harmonic oscillator

1. We use the ladder operators for the harmonic oscillator:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\Omega}}(\hat{a}(t) + \hat{a}^\dagger(t)) \text{ and } \hat{a}(t) = \hat{a}(0)e^{-i\Omega t}. \tag{26}$$

Rewriting the exponential function as a combination of cosine and sine functions we obtain the required expression for the autocorrelation function.

2.

$$\begin{aligned}\langle \hat{x}(0)\hat{p}(0)\rangle &= \mathrm{tr}\left[\hat{\rho}\hat{x}(0)\hat{p}(0)\right] && (27)\\ &= \frac{i\hbar}{2}\frac{1}{Z}\sum_n e^{-\beta\hbar\Omega n}\langle n|(\hat{a}+\hat{a}^\dagger)(\hat{a}-\hat{a}^\dagger)|n\rangle && (28)\\ &= \frac{i\hbar}{2} && (29)\end{aligned}$$

The second equality follows from the commutation relation.

[1] Carmichael, "Statistical Methods in Quantum Optics 1", section 1.5.

3. Using the same technique as in the last point (trace of the density matrix)

$$\langle \hat{x}(0)\hat{x}(0)\rangle = \frac{\hbar}{m\Omega}(\bar{n} + \frac{1}{2}) \, . \tag{30}$$

   Using trigonometric to exponential identities we obtain the desired result.

4. The spectral density is defined as

$$S_{xx}[\omega] = \int_{-\infty}^{\infty} \mathrm{d}t \, e^{i\omega t} \, \langle \hat{x}(t)\hat{x}(0)\rangle \ . \tag{31}$$

   Calculating this integral leads to

$$S_{xx}[\omega] = \frac{\pi\hbar}{m\Omega} \left(\bar{n}\delta(\omega + \Omega) + (\bar{n} + 1)\delta(\omega - \Omega)\right) \ , \tag{32}$$

   which is obviously not symmetric in frequency. In the high temperature limit $\bar{n} \to \frac{k_B T}{\hbar\omega} \gg 1$ so that $\bar{n} + 1 \to \bar{n}$ and the spectral density becomes symmetric.