CHAPTER 8

Quantum theory of damping – density operator and wave function approach

In many problems in quantum optics, damping plays an important role. These include, for example, the decay of an atom in an excited state to a lower state and the decay of the radiation field inside a cavity with partially transparent mirrors. In general, damping of a *system* is described by its interaction with a *reservoir* with a large number of degrees of freedom. We are interested, however, in the evolution of the variables associated with the system only. This requires us to obtain the equations of motion for the system of interest only after tracing over the reservoir variables. There are several different approaches to deal with this problem.

In this chapter, we present a theory of damping based on the density operator in which the reservoir variables are eliminated by using the *reduced* density operator for the system in the Schrödinger (or interaction) picture. We also present a 'quantum jump' approach to damping. In the next chapter, the damping of the system will be considered using the noise operator method in the Heisenberg picture.

An insight into the damping mechanism is obtained by considering the decay of an atom in an excited state inside a cavity. The atom may be considered as a single system coupled to the radiation field inside the cavity. Even in the absence of photons in the cavity, there are quantum fluctuations associated with the vacuum state. As discussed in Chapter 1, the field may be visualized as a large number of harmonic oscillators, one for each mode of the cavity. As the size of the cavity increases, the mode density increases, and, in free space, we get a continuum of modes. There is therefore a "cavity mode" which is resonant with the atomic transition.

We can also visualize the atom as an oscillator, with the excited atom corresponding to an oscillator in the excited state. The coupling of the atom to a large number of oscillators (associated with the large number of field modes) leads to decay. That is energy initially in the atom will distribute itself among damping oscillators, thus causing the decay of the atom to a lower energy state.

The dissipation is accompanied by fluctuations. We shall encounter this aspect of the damping mechanism, more formally put in the form of the so-called fluctuation–dissipation theorem, in the systems studied in this and the following chapters. We now start with a general reservoir theory before considering the atom and field damping by a reservoir of harmonic oscillator (bosonic) modes.

8.1 General reservoir theory

We consider in general a system denoted by S interacting with a reservoir denoted by R. The combined density operator is denoted by ρ_{SR}. The reduced density operator for the system ρ_S is obtained by taking a trace over the reservoir coordinates, i.e.,

$$\rho_S = \mathrm{Tr}_R(\rho_{SR}). \tag{8.1.1}$$

We assume that the system–reservoir interaction energy is given by $\mathscr{V}(t)$. The equation of motion for ρ_{SR} is then given by

$$i\hbar\dot{\rho}_{SR} = [\mathscr{V}(t), \rho_{SR}(t)]. \tag{8.1.2}$$

This equation can be formally integrated, and we obtain

$$\rho_{SR}(t) = \rho_{SR}(t_i) - \frac{i}{\hbar}\int_{t_i}^{t} [\mathscr{V}(t'), \rho_{SR}(t')]dt'. \tag{8.1.3}$$

Here t_i is an initial time when the interaction starts. On substituting $\rho_{SR}(t)$ back into Eq. (8.1.2), we find the equation of motion

$$\dot{\rho}_{SR} = -\frac{i}{\hbar}[\mathscr{V}(t), \rho_{SR}(t_i)] - \frac{1}{\hbar^2}\int_{t_i}^{t}[\mathscr{V}(t), [\mathscr{V}(t'), \rho_{SR}(t')]]dt'. \tag{8.1.4}$$

If the interaction energy $\mathscr{V}(t)$ is zero, the system and reservoir are independent and the density operator ρ_{SR} would factor as a direct product $\rho_{SR}(t) = \rho_S(t) \otimes \rho_R(t_i)$ where we assume the reservoir at equilibrium. Since $\mathscr{V}$ is small, we look for a solution of Eq. (8.1.4) of the form

$$\rho_{SR}(t) = \rho_S(t) \otimes \rho_R(t_i) + \rho_c(t), \tag{8.1.5}$$

where $\rho_c(t)$ is of higher order in $\mathscr{V}$. In order to satisfy (8.1.1), we require

$$\mathrm{Tr}_R[\rho_c(t)] = 0. \tag{8.1.6}$$

If we substitute for $\rho_{SR}(t)$ from Eq. (8.1.5) into the integrand of (8.1.4), and retain terms up to order $\mathscr{V}^2$, we have

$$\begin{aligned}\dot{\rho}_S = &-\frac{i}{\hbar}\mathrm{Tr}_R[\mathscr{V}(t), \rho_S(t_i) \otimes \rho_R(t_i)] \\ &-\frac{1}{\hbar^2}\mathrm{Tr}_R \int_{t_i}^{t} \left[\mathscr{V}(t), [\mathscr{V}(t'), \rho_S(t') \otimes \rho_R(t_i)]\right] dt'.\end{aligned} \tag{8.1.7}$$

The reduced density operator $\rho_S(t)$, which determines the statistical properties of the system, depends on its past history from $t = t_i$ to t'. This can be seen in Eq. (8.1.7) as $\rho_S(t')$ occurs in the integrand. However, the reservoir is typically an extended open system having many degrees of freedom. Moreover, as is shown by specific example in the next section, the large number of reservoir degrees of freedom (modes, photons, etc.) leads to a delta function $\delta(t - t')$. Hence, the system density matrix $\rho_S(t')$ can be replaced by $\rho_S(t)$ and the process is said to be *Markovian*. This is a reasonable assumption since damping destroys memory of the past. Equation (8.1.7) now becomes

$$\begin{aligned}\dot{\rho}_S = &-\frac{i}{\hbar}\mathrm{Tr}_R[\mathscr{V}(t), \rho_S(t_i) \otimes \rho_R(t_i)] \\ &-\frac{1}{\hbar^2}\mathrm{Tr}_R \int_{t_i}^{t} [\mathscr{V}(t), [\mathscr{V}(t'), \rho_S(t) \otimes \rho_R(t_i)]]dt'.\end{aligned} \tag{8.1.8}$$

This is a valid equation for a system represented by ρ_S interacting with a reservoir represented by ρ_R. In the next sections, we consider several examples of the system–reservoir interaction.

8.2 Atomic decay by thermal and squeezed vacuum reservoirs

The decay of an atom in an excited state may be understood from a simple model in which the atom is coupled to a reservoir of simple harmonic oscillators. In a very similar manner, the decay of the radiation field inside a cavity may be described by a model in which the mode of the field of interest is coupled to a whole set of reservoir modes. Such problems are of interest not only in maser and laser physics, but also in the quantum theory of passive interferometers such as those used in the detection of gravitational waves.

We first consider the radiative decay of a two-level atom damped by a reservoir of simple harmonic oscillators described by annihilation (and creation) operators $b_{\mathbf{k}}$ (and $b_{\mathbf{k}}^{\dagger}$) and density distributed frequencies $\nu_k = ck$. In the interaction picture and the rotating-wave approximation, the Hamiltonian is simply

$$\mathscr{V}(t) = \hbar \sum_{\mathbf{k}} g_{\mathbf{k}} \left[b_{\mathbf{k}}^{\dagger} \sigma_{-} e^{-i(\omega-\nu_k)t} + \sigma_{+} b_{\mathbf{k}} e^{i(\omega-\nu_k)t} \right], \tag{8.2.1}$$

where $\sigma_{-} = |b\rangle\langle a|$ and $\sigma_{+} = |a\rangle\langle b|$ in terms of the excited ($|a\rangle$) and ground ($|b\rangle$) states. The system now corresponds to the two-level atom ($\rho_S \equiv \rho_{\text{atom}}$). On inserting the interaction energy $\mathscr{V}$ (Eq. (8.2.1)) into the equation of motion (8.1.7) for $\rho_S \equiv \rho_{\text{atom}}$, we obtain

$$\begin{aligned}
\dot{\rho}_{\text{atom}} = & -i \sum_{\mathbf{k}} g_{\mathbf{k}} \langle b_{\mathbf{k}}^{\dagger} \rangle [\sigma_{-}, \rho_{\text{atom}}(t_i)] e^{-i(\omega-\nu_k)t} \\
& - \int_{t_i}^{t} dt' \sum_{\mathbf{k},\mathbf{k}'} g_{\mathbf{k}} g_{\mathbf{k}'} \{ [\sigma_{-}\sigma_{-}\rho_{\text{atom}}(t') - 2\sigma_{-}\rho_{\text{atom}}(t')\sigma_{-} \\
& \qquad + \rho_{\text{atom}}(t')\sigma_{-}\sigma_{-}] \\
& \times e^{-i(\omega-\nu_k)t - i(\omega-\nu_{k'})t'} \langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'}^{\dagger} \rangle + [\sigma_{-}\sigma_{+}\rho_{\text{atom}}(t') - \sigma_{+}\rho_{\text{atom}}(t')\sigma_{-}] \\
& \times e^{-i(\omega-\nu_k)t + i(\omega-\nu_{k'})t'} \langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'} \rangle + [\sigma_{+}\sigma_{-}\rho_{\text{atom}}(t') - \sigma_{-}\rho_{\text{atom}}(t')\sigma_{+}] \\
& \times e^{i(\omega-\nu_k)t - i(\omega-\nu_{k'})t'} \langle b_{\mathbf{k}} b_{\mathbf{k}'}^{\dagger} \rangle \} + \text{H.c.},
\end{aligned} \tag{8.2.2}$$

where the expectation values refer to the initial state of the reservoir. At this point we choose a particular model for the state of the reservoir.

8.2.1 *Thermal reservoir*

As a first example, we assume that the reservoir variables are distributed in the uncorrelated thermal equilibrium mixture of states. The reservoir reduced density operator is the multi-mode extension of the thermal operator, namely,

$$\rho_R = \prod_{\mathbf{k}} \left[1 - \exp\left(-\frac{\hbar \nu_k}{k_B T} \right) \right] \exp\left(-\frac{\hbar \nu_k b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}}{k_B T} \right), \tag{8.2.3}$$

where k_B is the Boltzmann constant and T is the temperature. It can be shown easily that

$$\langle b_{\mathbf{k}} \rangle = \langle b_{\mathbf{k}}^{\dagger} \rangle = 0, \tag{8.2.4a}$$

$$\langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'} \rangle = \bar{n}_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'}, \tag{8.2.4b}$$

$$\langle b_{\mathbf{k}} b_{\mathbf{k}'}^{\dagger} \rangle = (\bar{n}_{\mathbf{k}} + 1) \delta_{\mathbf{k}\mathbf{k}'}, \tag{8.2.4c}$$

$$\langle b_{\mathbf{k}} b_{\mathbf{k}'} \rangle = \langle b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'}^{\dagger} \rangle = 0, \tag{8.2.4d}$$

where the thermal average boson number

$$\bar{n}_{\mathbf{k}} = \frac{1}{\exp\left(\frac{\hbar\nu_k}{k_B T}\right) - 1}. \tag{8.2.5}$$

On substituting for the various expectation values from Eqs. (8.2.4 into Eq. (8.2.2), we obtain

$$\begin{aligned}\dot{\rho}_{\text{atom}} &= -\int_{t_i}^{t} dt' \sum_{\mathbf{k}} g_{\mathbf{k}}^2 \{[\sigma_-\sigma_+\rho_{\text{atom}}(t') - \sigma_+\rho_{\text{atom}}(t')\sigma_-] \\ &\quad \bar{n}_{\mathbf{k}} e^{-i(\omega-\nu_k)(t-t')} \\ &\quad + [\sigma_+\sigma_-\rho_{\text{atom}}(t') - \sigma_-\rho_{\text{atom}}(t')\sigma_+](\bar{n}_{\mathbf{k}}+1)e^{i(\omega-\nu_k)(t-t')}\} + \text{H.c.}\end{aligned} \tag{8.2.6}$$

We now carry out the same procedure as was used in the Weisskopf–Wigner theory of spontaneous emission.

The sum over **k** may be replaced by an integral through the standard prescription (Eq. (6.3.9))

$$\sum_{\mathbf{k}} \longrightarrow 2\frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2, \tag{8.2.7}$$

where V is the quantization volume. The integrations in Eq. (8.2.6) can be carried out in the Weisskopf–Wigner approximation as discussed in Section 6.3. In this way, we encounter integrals of the form (6.3.12). We thus find for the reduced density operator ρ_{atom}

$$\begin{aligned}\dot{\rho}_{\text{atom}}(t) &= -\bar{n}_{\text{th}}\frac{\Gamma}{2}[\sigma_-\sigma_+\rho_{\text{atom}}(t) - \sigma_+\rho_{\text{atom}}(t)\sigma_-] \\ &\quad -(\bar{n}_{\text{th}}+1)\frac{\Gamma}{2}[\sigma_+\sigma_-\rho_{\text{atom}}(t) - \sigma_-\rho_{\text{atom}}(t)\sigma_+] + \text{H.c.},\end{aligned} \tag{8.2.8}$$

where $\bar{n}_{\text{th}} \equiv \bar{n}_{k_0}$ $(k_0 = \omega/c)$ and

$$\Gamma = \frac{1}{4\pi\epsilon_0}\frac{4\omega^3\wp_{ab}^2}{3\hbar c^3} \tag{8.2.9}$$

is the atomic decay rate which is identical to the decay constant (Eq. (6.3.14)) derived in the Weisskopf–Wigner theory of spontaneous emission. In deriving Eq. (8.2.8) we substituted the value of $g_{\mathbf{k}}$ from Eq. (6.1.8).

The equations of motion for the atomic density matrix elements can now be obtained from Eq. (8.2.8):

$$\dot{\rho}_{aa} = \langle a|\dot{\rho}_{\text{atom}}|a\rangle$$
$$= -(\bar{n}_{\text{th}} + 1)\Gamma\rho_{aa} + \bar{n}_{\text{th}}\Gamma\rho_{bb}, \tag{8.2.10a}$$
$$\dot{\rho}_{ab} = \dot{\rho}^*_{ba} = -\left(\bar{n}_{\text{th}} + \frac{1}{2}\right)\Gamma\rho_{ab}, \tag{8.2.10b}$$
$$\dot{\rho}_{bb} = -\bar{n}_{\text{th}}\Gamma\rho_{bb} + (\bar{n}_{\text{th}} + 1)\Gamma\rho_{aa}. \tag{8.2.10c}$$

It may be noted that $\dot{\rho}_{aa} + \dot{\rho}_{bb} = 0$. This is due to the fact that we are considering the decay from the upper level $|a\rangle$ to the lower level $|b\rangle$ only. The conservation of probability therefore implies $\rho_{aa} + \rho_{bb} = 1$. This situation is different from that discussed in Section 5.3, where atomic levels $|a\rangle$ and $|b\rangle$ decayed to some other levels via nonradiating transitions. For zero temperature ($\bar{n}_{\text{th}} = 0$), these equations simplify to

$$\dot{\rho}_{aa} = -\Gamma\rho_{aa}, \tag{8.2.11a}$$
$$\dot{\rho}_{ab} = -\frac{\Gamma}{2}\rho_{ab}, \tag{8.2.11b}$$
$$\dot{\rho}_{bb} = \Gamma\rho_{aa}. \tag{8.2.11c}$$

Equation (8.2.11a) is just the Weisskopf–Wigner result (6.3.15).

8.2.2 Squeezed vacuum reservoir

For our second example, we consider the situation where the atom is coupled to a squeezed vacuum field reservoir. The reservoir reduced density operator is given by

$$\rho_R = |\xi\rangle\langle\xi|$$
$$= \prod_{\mathbf{k}} S_{\mathbf{k}}(\xi)|0_{\mathbf{k}}\rangle\langle 0_{\mathbf{k}}|S^\dagger_{\mathbf{k}}(\xi), \tag{8.2.12}$$

where the squeeze operator (see Eq. (2.8.9) with $b_{\mathbf{k}} \equiv b(ck)$, etc.) is

$$S_{\mathbf{k}}(\xi) = \exp\left(\xi^* b_{\mathbf{k}_0+\mathbf{k}} b_{\mathbf{k}_0-\mathbf{k}} - \xi b^\dagger_{\mathbf{k}_0+\mathbf{k}} b^\dagger_{\mathbf{k}_0-\mathbf{k}}\right), \tag{8.2.13}$$

with $\xi = r\exp(i\theta)$, r being the squeeze parameter and θ being the reference phase for the squeezed field. A multi-mode squeezed field is not just a product of independently squeezed modes, rather there are correlations between modes symmetrically placed about the central, resonant frequency $\nu = ck_0$ of the squeezing device. Following the method used to derive Eqs. (2.7.6) and (2.7.7), we obtain

$$S^{\dagger}_{\mathbf{k}-\mathbf{k}_0} b_{\mathbf{k}} S_{\mathbf{k}-\mathbf{k}_0} = b_{\mathbf{k}} \cosh(r) - b^{\dagger}_{2\mathbf{k}_0-\mathbf{k}} e^{i\theta} \sinh(r), \tag{8.2.14a}$$

$$S^{\dagger}_{\mathbf{k}-\mathbf{k}_0} b^{\dagger}_{\mathbf{k}} S_{\mathbf{k}-\mathbf{k}_0} = b^{\dagger}_{\mathbf{k}} \cosh(r) - b_{2\mathbf{k}_0-\mathbf{k}} e^{-i\theta} \sinh(r). \tag{8.2.14b}$$

Similar expressions exist for $S^{\dagger}_{\mathbf{k}_0-\mathbf{k}} b_{\mathbf{k}} S_{\mathbf{k}_0-\mathbf{k}}$ and $S^{\dagger}_{\mathbf{k}_0-\mathbf{k}} b^{\dagger}_{\mathbf{k}} S_{\mathbf{k}_0-\mathbf{k}}$. The calculation of the expectation values, such as $\langle b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}'}\rangle$, may therefore be simplified by writing

$$\langle b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}'}\rangle = \prod_{\mathbf{q}} \langle 0_{\mathbf{q}}|S^{\dagger}_{\mathbf{q}} b^{\dagger}_{\mathbf{k}} S_{\mathbf{q}} S^{\dagger}_{\mathbf{q}} b_{\mathbf{k}'} S_{\mathbf{q}}|0_{\mathbf{q}}\rangle. \tag{8.2.15}$$

It follows that

$$\langle b_{\mathbf{k}}\rangle = \langle b^{\dagger}_{\mathbf{k}}\rangle = 0, \tag{8.2.16a}$$

$$\langle b^{\dagger}_{\mathbf{k}} b_{\mathbf{k}'}\rangle = \sinh^2(r)\delta_{\mathbf{k}\mathbf{k}'}, \tag{8.2.16b}$$

$$\langle b_{\mathbf{k}} b^{\dagger}_{\mathbf{k}'}\rangle = \cosh^2(r)\delta_{\mathbf{k}\mathbf{k}'}, \tag{8.2.16c}$$

$$\langle b_{\mathbf{k}} b_{\mathbf{k}'}\rangle = -e^{i\theta} \sinh(r)\cosh(r)\delta_{\mathbf{k}',2\mathbf{k}_0-\mathbf{k}}, \tag{8.2.16d}$$

$$\langle b^{\dagger}_{\mathbf{k}} b^{\dagger}_{\mathbf{k}'}\rangle = -e^{-i\theta} \sinh(r)\cosh(r)\delta_{\mathbf{k}',2\mathbf{k}_0-\mathbf{k}}. \tag{8.2.16e}$$

On substituting Eqs. (8.2.16a-8.2.16e) into Eq. (8.2.2) and proceeding as in the derivation of Eq. (8.2.8), we obtain

$$\begin{aligned}\dot{\rho}_{\text{atom}} = &-\frac{\Gamma}{2}\cosh^2(r)(\sigma_+\sigma_-\rho_{\text{atom}} - 2\sigma_-\rho_{\text{atom}}\sigma_+ + \rho_{\text{atom}}\sigma_+\sigma_-) \\ &-\frac{\Gamma}{2}\sinh^2(r)(\sigma_-\sigma_+\rho_{\text{atom}} - 2\sigma_+\rho_{\text{atom}}\sigma_- + \rho_{\text{atom}}\sigma_-\sigma_+) \\ &-\Gamma e^{-i\theta}\sinh(r)\cosh(r)\sigma_-\rho_{\text{atom}}\sigma_- \\ &-\Gamma e^{i\theta}\sinh(r)\cosh(r)\sigma_+\rho_{\text{atom}}\sigma_+.\end{aligned} \tag{8.2.17}$$

In deriving Eq. (8.2.17) we used $\sigma_-\sigma_- = \sigma_+\sigma_+ = 0$.

From Eq. (8.2.17), equations of motion for the expectation value of the operators $\sigma_x = (\sigma_- + \sigma_+)/2$, $\sigma_y = (\sigma_- - \sigma_+)/2i$, and $\sigma_z = (2\sigma_+\sigma_- - 1)$ are

$$\langle\dot{\sigma}_x\rangle = -\frac{\Gamma}{2}e^{2r}\langle\sigma_x\rangle, \tag{8.2.18a}$$

$$\langle\dot{\sigma}_y\rangle = -\frac{\Gamma}{2}e^{-2r}\langle\sigma_y\rangle, \tag{8.2.18b}$$

$$\langle\dot{\sigma}_z\rangle = -\Gamma[2\sinh^2(r) + 1]\langle\sigma_z\rangle - \Gamma = -\Gamma_z\langle\sigma_z\rangle - \Gamma, \tag{8.2.18c}$$

where $\Gamma_z = \Gamma[2\sinh^2(r) + 1]$ and we have chosen the phase $\theta = 0$. It is therefore clear that a squeezed vacuum reservoir leads to a phase sensitive decay of the atom. The in-phase and in-quadrature components, $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$, of the atomic dipole moment decay at different rates depending on its initial phase relative to the phase θ of the squeezed vacuum.

8.3 Field damping

We may apply the method developed in the last section to the decay of a mode of the electromagnetic field of frequency ν inside a cavity. Instead of Eq. (8.2.1), we now use an interaction Hamiltonian of the form

$$\mathscr{V} = \hbar \sum_{\mathbf{k}} g_{\mathbf{k}} [b_{\mathbf{k}}^{\dagger} a e^{-i(\nu-\nu_k)t} + a^{\dagger} b_{\mathbf{k}} e^{i(\nu-\nu_k)t}], \qquad (8.3.1)$$

where a (and $a^{\dagger}$) are the destruction (and creation) operators of the mode of interest. The operators $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\dagger}$ represent modes of the reservoir which damp the field. For transmission losses they actually represent the field outside the cavity.

The equation of motion for the reduced density operator for the field can now easily be obtained, since the calculation exactly parallels the one for the atomic system discussed in the last section. This is done by replacing σ_- and σ_+ by the field operators a and $a^{\dagger}$, respectively.

When the modes $b_{\mathbf{k}}$ are initially in the thermal equilibrium mixture of states (8.2.3), the result is

$$\begin{aligned}\dot{\rho} = &-\frac{\mathscr{C}}{2}\bar{n}_{\text{th}}(aa^{\dagger}\rho - 2a^{\dagger}\rho a + \rho a a^{\dagger}) \\ &-\frac{\mathscr{C}}{2}(\bar{n}_{\text{th}} + 1)(a^{\dagger}a\rho - 2a\rho a^{\dagger} + \rho a^{\dagger} a),\end{aligned} \qquad (8.3.2)$$

where, as before, $\mathscr{C}$ is the decay constant and $\bar{n}_{\text{th}} = \bar{n}_{\mathbf{k}_0}$ is the mean number of quanta (at frequency ν) in the thermal reservoir. Here ρ denotes the reduced density operator for the field. In particular, at zero temperature ($\bar{n}_{\text{th}} = 0$),

$$\dot{\rho} = -\frac{\mathscr{C}}{2}(a^{\dagger}a\rho - 2a\rho a^{\dagger} + \rho a^{\dagger} a). \qquad (8.3.3)$$

If all the losses are transmission losses, $\mathscr{C}$ may be related to the quality factor Q of the cavity by $\mathscr{C} = \nu/Q$.

When the modes $b_{\mathbf{k}}$ are initially in a squeezed vacuum (Eq. (8.2.12)), the resulting equation of motion for the reduced density matrix ρ is

$$\begin{aligned}\dot{\rho} = &-\frac{\mathscr{C}}{2}(N+1)(a^{\dagger}a\rho - 2a\rho a^{\dagger} + \rho a^{\dagger} a) \\ &-\frac{\mathscr{C}}{2}N(aa^{\dagger}\rho - 2a^{\dagger}\rho a + \rho a a^{\dagger}) \\ &+\frac{\mathscr{C}}{2}M(aa\rho - 2a\rho a + \rho aa) \\ &+\frac{\mathscr{C}}{2}M^{*}(a^{\dagger}a^{\dagger}\rho - 2a^{\dagger}\rho a^{\dagger} + \rho a^{\dagger}a^{\dagger}),\end{aligned} \qquad (8.3.4)$$

where $N = \sinh^2(r)$ and $M = \cosh(r)\sinh(r)\exp(-i\theta)$. This equation describes, for instance, the evolution of the field in a cavity coupled through a partially transmitting mirror to an outside field which is in a squeezed vacuum state. The equation of motion for the thermal reservoir (Eq. (8.3.2)) can be recovered from Eq. (8.3.4) by the substitutions $N \to \bar{n}_{\text{th}}$, $M \to 0$. The parameters N and M are however related to each other via the equation $|M| = [N(N+1)]^{1/2}$ for a squeezed vacuum reservoir.

8.4 Fokker–Planck equation

A particularly interesting representation into which the density operator equation of motion can be transformed is the coherent state representation or P-representation discussed in Chapter 3. In this section, we derive an equation of motion for the P-representation corresponding to Eq. (8.3.2) for the density operator for a harmonic oscillator mode damped by a thermal bath full of harmonic oscillators. The resulting equation will have the form of a Fokker–Planck equation. The solution of this equation will reveal some interesting features about the temporal evolution of the field distribution.

We substitute the P-representation, see Eq. (3.1.16),

$$\rho = \int P(\alpha, \alpha^*, t)|\alpha\rangle\langle\alpha| d^2\alpha \tag{8.4.1}$$

into Eq. (8.3.2) and the resulting equation is

$$\begin{aligned}\int \dot{P}(\alpha, \alpha^*, t)|\alpha\rangle\langle\alpha| d^2\alpha = &-\frac{\mathscr{C}}{2}\bar{n}_{\text{th}} \int P(\alpha, \alpha^*, t)(aa^\dagger|\alpha\rangle\langle\alpha| \\ &-2a^\dagger|\alpha\rangle\langle\alpha|a + |\alpha\rangle\langle\alpha|aa^\dagger)d^2\alpha \\ &-\frac{\mathscr{C}}{2}(\bar{n}_{\text{th}} + 1)\int P(\alpha, \alpha^*, t)(a^\dagger a|\alpha\rangle\langle\alpha| \\ &-2a|\alpha\rangle\langle\alpha|a^\dagger + |\alpha\rangle\langle\alpha|a^\dagger a)d^2\alpha.\end{aligned} \tag{8.4.2}$$

It follows from

$$a^\dagger|\alpha\rangle\langle\alpha| = \left(\frac{\partial}{\partial\alpha} + \alpha^*\right)|\alpha\rangle\langle\alpha|, \tag{8.4.3a}$$

$$a|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha|, \tag{8.4.3b}$$

$$|\alpha\rangle\langle\alpha|a^\dagger = \alpha^*|\alpha\rangle\langle\alpha|, \tag{8.4.3c}$$

$$|\alpha\rangle\langle\alpha|a = \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)|\alpha\rangle\langle\alpha|, \tag{8.4.3d}$$

that

$$
\begin{aligned}
& aa^{\dagger}|\alpha\rangle\langle\alpha| - 2a^{\dagger}|\alpha\rangle\langle\alpha|a + |\alpha\rangle\langle\alpha|aa^{\dagger} \\
&= \left[\left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\alpha - 2\left(\frac{\partial}{\partial\alpha} + \alpha^*\right)\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\right. \\
&\quad \left. + \left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\alpha^*\right]|\alpha\rangle\langle\alpha| \\
&= -\left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*} + 2\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|,
\end{aligned}
\tag{8.4.4}
$$

and

$$
\begin{aligned}
& a^{\dagger}a|\alpha\rangle\langle\alpha| - 2a|\alpha\rangle\langle\alpha|a^{\dagger} + |\alpha\rangle\langle\alpha|a^{\dagger}a \\
&= \left[\alpha\left(\frac{\partial}{\partial\alpha} + \alpha^*\right) - 2|\alpha|^2 + \alpha^*\left(\frac{\partial}{\partial\alpha^*} + \alpha\right)\right]|\alpha\rangle\langle\alpha| \\
&= \left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|.
\end{aligned}
\tag{8.4.5}
$$

We now substitute Eqs. (8.4.4) and (8.4.5) into Eq. (8.4.2) and integrate the result by parts. In doing so we encounter the integral

$$
\begin{aligned}
& \int P(\alpha,\alpha^*,t)\left(\alpha\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha|\right)d^2\alpha \\
&= \alpha P(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha|\Big|_{-\infty}^{\infty} - \int\left[\frac{\partial}{\partial\alpha}\alpha P(\alpha,\alpha^*,t)\right]|\alpha\rangle\langle\alpha|d^2\alpha.
\end{aligned}
\tag{8.4.6}
$$

Since the distribution vanishes at the infinite limits, Eq. (8.4.6) becomes

$$
\int P(\alpha,\alpha^*,t)\left(\alpha\frac{\partial}{\partial\alpha}|\alpha\rangle\langle\alpha|\right)d^2\alpha = -\int\left[\frac{\partial}{\partial\alpha}\alpha P(\alpha,\alpha^*,t)\right]|\alpha\rangle\langle\alpha|d^2\alpha.
\tag{8.4.7}
$$

Similarly

$$
\int P(\alpha,\alpha^*,t)\left(\frac{\partial^2}{\partial\alpha\partial\alpha^*}|\alpha\rangle\langle\alpha|\right)d^2\alpha = \int\left[\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha,\alpha^*,t)\right]|\alpha\rangle\langle\alpha|d^2\alpha.
\tag{8.4.8}
$$

Then we have from Eq. (8.4.2)

$$
\begin{aligned}
\int \dot{P}(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha|d^2\alpha &= \frac{\mathscr{C}}{2}\int\left[\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* + 2\bar{n}_{\text{th}}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\right. \\
&\quad \left. \times P(\alpha,\alpha^*,t)\right]|\alpha\rangle\langle\alpha|d^2\alpha.
\end{aligned}
\tag{8.4.9}
$$

It follows on identifying the coefficients of $|\alpha\rangle\langle\alpha|$ in the integrands that the equation of motion for $P(\alpha,\alpha^*,t)$ is

$$\dot{P} = \frac{\mathscr{C}}{2}\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right)P + \mathscr{C}\bar{n}_{\text{th}}\frac{\partial^2 P}{\partial\alpha\partial\alpha^*}. \tag{8.4.10}$$

This is the Fokker–Planck equation for the P-representation.

Next we find a solution of the Fokker–Planck equation. We assume that the field is initially in a coherent state $|\alpha_0\rangle$, i.e.,

$$P(\alpha,\alpha^*,0) = \delta^{(2)}(\alpha - \alpha_0). \tag{8.4.11}$$

In the Gaussian representation of the δ-function,

$$P(\alpha,\alpha^*,0) = \lim_{\epsilon\to 0}\frac{1}{\pi\epsilon}\exp\left(\frac{-|\alpha-\alpha_0|^2}{\epsilon}\right). \tag{8.4.12}$$

We therefore seek a solution of Eq. (8.4.10) in the form

$$P(\alpha,\alpha^*,t) = \exp[-a(t) + b(t)\alpha + c(t)\alpha^* - d(t)\alpha\alpha^*], \tag{8.4.13}$$

subject to the initial conditions

$$a(0) = \frac{|\alpha_0|^2}{\epsilon} + \ln(\pi\epsilon), \tag{8.4.14a}$$

$$b(0) = \frac{\alpha_0^*}{\epsilon}, \tag{8.4.14b}$$

$$c(0) = \frac{\alpha_0}{\epsilon}, \tag{8.4.14c}$$

$$d(0) = \frac{1}{\epsilon}. \tag{8.4.14d}$$

On substituting expression (8.4.13) for $P(\alpha,\alpha^*,t)$ into Eq. (8.4.10) and carrying out the necessary t and α differentiations, we obtain

$$\begin{aligned}-\dot{a} + \dot{b}\alpha + \dot{c}\alpha^* - \dot{d}|\alpha|^2 = \mathscr{C}\Bigg[1 + \bar{n}_{\text{th}}(bc-d) + \left(\frac{b}{2} - \bar{n}_{\text{th}}bd\right)\alpha \\ + \left(\frac{c}{2} - \bar{n}_{\text{th}}cd\right)\alpha^* - (d - \bar{n}_{\text{th}}d^2)|\alpha|^2\Bigg].\end{aligned} \tag{8.4.15}$$

A comparison of the terms proportional to $|\alpha|^2$, α^*, α, and unity lead to the following set of differential equations:

$$\dot{d} = \mathscr{C}(d - \bar{n}_{\text{th}}d^2), \tag{8.4.16a}$$

$$\dot{c} = \mathscr{C}\left(\frac{c}{2} - \bar{n}_{\text{th}}cd\right), \tag{8.4.16b}$$

$$\dot{b} = \mathscr{C}\left(\frac{b}{2} - \bar{n}_{\text{th}}bd\right), \tag{8.4.16c}$$

$$\dot{a} = -\mathscr{C}[1 + \bar{n}_{\text{th}}(bc - d)]. \tag{8.4.16d}$$

The solution of these equations subject to the initial conditions (8.4.14a)–(8.4.14d) is given by

$$d(t) = \frac{1}{\bar{n}_{\rm th}(1 - e^{-\mathscr{C}t}) + \epsilon e^{-\mathscr{C}t}}, \tag{8.4.17a}$$

$$c(t) = \frac{\alpha_0 e^{-\mathscr{C}t/2}}{\bar{n}_{\rm th}(1 - e^{-\mathscr{C}t}) + \epsilon e^{-\mathscr{C}t}}, \tag{8.4.17b}$$

$$b(t) = \frac{\alpha_0^* e^{-\mathscr{C}t/2}}{\bar{n}_{\rm th}(1 - e^{-\mathscr{C}t}) + \epsilon e^{-\mathscr{C}t}}, \tag{8.4.17c}$$

$$a(t) = \frac{|\alpha_0|^2 e^{-\mathscr{C}t}}{\bar{n}_{\rm th}(1 - e^{-\mathscr{C}t}) + \epsilon e^{-\mathscr{C}t}} + \ln\left\{\pi\left[\bar{n}_{\rm th}\left(1 - e^{-\mathscr{C}t}\right) + \epsilon e^{-\mathscr{C}t}\right]\right\}. \tag{8.4.17d}$$

A substitution of these solutions into Eq. (8.4.13) results in the Gaussian form for $P(\alpha, \alpha^*, t)$:

$$P(\alpha, \alpha^*, t) = \frac{1}{\pi D(t)} \exp\left[-\frac{|\alpha - \alpha_0 U(t)|^2}{D(t)}\right], \tag{8.4.18}$$

where

$$D(t) = \bar{n}_{\rm th}(1 - e^{-\mathscr{C}t}) \tag{8.4.19}$$

is the dispersion of the Gaussian function about its mean value

$$\alpha_0 U(t) = \alpha_0 e^{-\mathscr{C}t/2 - i\nu t}. \tag{8.4.20}$$

In Eq. (8.4.20), we have included the factor $\exp(-i\nu t)$ by going back from the interaction picture to the Schrödinger picture.

The dispersion $D(t)$ increases from the initial value zero, while the center of the Gaussian distribution circles about on the exponential spiral given by Eq. (8.4.20). This is shown in Fig. 8.1 where the P-representation is plotted as a function of complex amplitude α. When the time t is much greater than the damping time, $\mathscr{C}^{-1}$, the field distribution comes to equilibrium with the heat bath oscillators. In the steady state, the dispersion has its limiting value $\bar{n}_{\rm th}$ and the Gaussian distribution is centered about the origin. Thus the field loses its initial excitation to the heat bath oscillators but acquires noise in the process of damping. This is a manifestation of the fluctuation–dissipation theorem, i.e., the dissipation via heat bath oscillators is accompanied by fluctuations. We will discuss it in the next chapter.

It is interesting to note that if we take the heat bath to be at zero temperature ($\bar{n}_{\rm th} = 0$), the dispersion $D(t)$ remains zero at all times and $P(\alpha, \alpha^*, t)$ always remains a δ-function, i.e.,

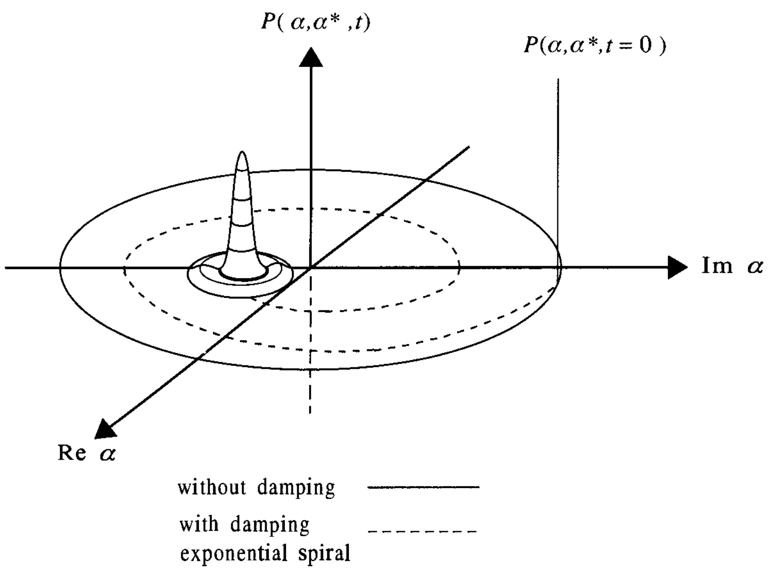

Fig. 8.1
The P-representation for the complex amplitude of a harmonic oscillator mode damped by a thermal bath. The harmonic oscillator mode starts at $t = 0$ in a pure coherent sate $|\alpha\rangle$ and the mean value of the amplitude moves on an exponential spiral decreasing steadily in modulus, while its dispersion increases.

$$P(\alpha, \alpha^*, t) = \delta^{(2)}[\alpha - \alpha_0 U(t)]. \tag{8.4.21}$$

The state of the field remains at all times in a pure coherent state. This form of dissipation is completely noise free.

8.5 The 'quantum jump' approach to damping

Historically, the notions of quantum jumps and instantaneous collapse of the wave function go back to the early days in which Einstein worried about outgoing spherical waves 'collapsing' when a photoelectron is detected; and the notion of Bohr concerning the emission of light when an atom 'jumped' between Bohr orbits.

However, with the coming of wave mechanics the whole question of quantum jumps took on a new perspective. Atomic transitions were 'induced' and one often encountered statements that 'there were no such a thing as quantum jumps'.

Recently, the work of Dehmelt and others clearly shows that sudden jumps are evident in many aspects of quantum optics, e.g., the spectacular work involving single ions in a Paul trap.

More recently a new 'quantum jump' approach to dissipation has developed, one can find names and concepts like: Monte Carlo simulation, quantum trajectories, collapse or reduction of the state vector,

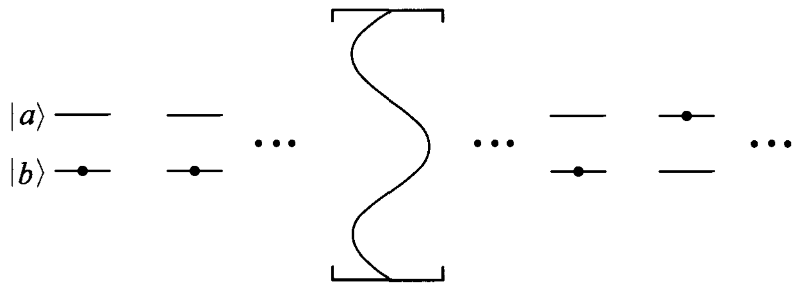

Fig. 8.2
Two-level atoms in their ground state $|b\rangle$ passing through a resonant cavity.

no count or 'null' measurement, and conditional density matrices. We will here give a short account of this interesting idea as it applies to damping or dissipation in quantum optics.

8.5.1 Conditional density matrices and the null measurement

In the previous sections of this chapter we have developed the theory of damping or dissipation in quantum mechanics from a density matrix perspective. The result is typically an expression of the form of (8.3.3) which describes the decay of a single mode of a resonant cavity at temperature $T = 0$. There we took the model of a large number of bath oscillators, e.g., phonons coupling energy out of the cavity mode. However the result, Eq. (8.3.3), is not specific to the model and we will here investigate the problem again using another model which will lead us naturally to a different point of view concerning dissipation processes.

Consider the model of Fig. 8.2 in which we are passing ground state atoms through a cavity which is resonant with the atoms, i.e., the Hamiltonian in the interaction picture is

$$\mathscr{V} = \hbar g(a^\dagger|b\rangle\langle a| + |a\rangle\langle b|a). \tag{8.5.1}$$

Consider the density matrix for the field at time $t+\tau$, $\rho(t+\tau)$, resulting from a ground state atom injected at time t, i.e.,

$$\begin{aligned}\rho(t+\tau) &= \mathrm{Tr}_{\mathrm{atom}}\left[e^{-i\mathscr{V}\tau/\hbar}\rho(t)\otimes|b\rangle\langle b|e^{i\mathscr{V}\tau/\hbar}\right]\\ &= \langle a|\rho_{\mathrm{atom-field}}(t+\tau)|a\rangle + \langle b|\rho_{\mathrm{atom-field}}(t+\tau)|b\rangle.\end{aligned} \tag{8.5.2}$$

It is natural to identify the two terms in (8.5.2) as 'conditional' density matrices, i.e.,

$$\begin{aligned}&\rho_a(t+\tau)\\ &= \text{conditional density matrix for field, atom excited}\\ &= \langle a|e^{-i\mathscr{V}\tau/\hbar}\rho(t)\otimes|b\rangle\langle b|e^{i\mathscr{V}\tau/\hbar}|a\rangle,\end{aligned} \tag{8.5.3a}$$

$$\rho_b(t+\tau)$$
$$= \text{conditional density matrix for field, atom not excited}$$
$$= \langle b|e^{-i\mathscr{V}\tau/\hbar}\rho(t)\otimes|b\rangle\langle b|e^{i\mathscr{V}\tau/\hbar}|b\rangle. \qquad (8.5.3b)$$

We may regard ρ_a and ρ_b as conditional density matrices corresponding to our having observed a count (excited atom) or no count (ground state atom) in our atomic beam. That is, the atomic beam serves two functions: it is a dissipation mechanism and it is also a kind of probe, or photodetector, of the field.

We proceed by noting that for small times τ, we may expand the $\exp(\pm i\mathscr{V}\tau/\hbar)$ factors and find

$$\rho_a(t+\tau) \cong g^2\tau^2 a\rho(t)a^\dagger, \qquad (8.5.4a)$$

$$\rho_b(t+\tau) \cong \rho(t) - \frac{1}{2}g^2\tau^2\left[a^\dagger a\rho(t) + \rho(t)a^\dagger a\right]$$
$$\cong e^{-R\tau a^\dagger a}\rho(t)e^{-R\tau a^\dagger a}, \qquad (8.5.4b)$$

where $R = g^2\tau/2$.

Now we make the key step. We let the time $\tau \to 0$ and make the ansatz that Eq. (8.5.4a) is to be associated with a 'quantum jump' of photoabsorption at time t. Then if we consider a process in which n counts are observed at times $t_1, t_2, ..., t_n$ with no counts in between these times, we have the conditional density matrix

$$\rho^{(n)} = \left[e^{-S(t-t_n)}ae^{-S(t_n-t_{n-1})}\ldots ae^{-S(t_2-t_1)}ae^{-St_1}\right.$$
$$\left.\times\rho(0)e^{-St_1}a^\dagger e^{-S(t_2-t_1)}a^\dagger e^{-S(t_n-t_{n-1})}a^\dagger e^{-S(t-t_n)}\right]$$
$$/\mathrm{Tr}, \qquad (8.5.5)$$

where $S = Ra^\dagger a$ and the trace factor in the denominator is the normalization factor. This may be simplified by taking account of the fact that, e.g.,

$$e^{-S(t_2-t_1)}ae^{-St_1} = e^{-Ra^\dagger a(t_2-t_1)}ae^{-Ra^\dagger at_1}$$
$$= e^{-Ra^\dagger a(t_2-t_1)}e^{-Ra^\dagger at_1}e^{Ra^\dagger at_1}ae^{-Ra^\dagger at_1}$$
$$= e^{-Ra^\dagger at_2}e^{-Rt_1}a, \qquad (8.5.6)$$

which may be used repeatedly to reduce Eq. (8.5.5) to the simple form

$$\rho^{(n)}(t) = \frac{e^{-Ra^\dagger at}a^n\rho(0)a^{\dagger n}e^{-Ra^\dagger at}}{\mathrm{Tr}[\rho(0)a^{\dagger n}e^{-2Ra^\dagger at}a^n]}, \qquad (8.5.7)$$

where the various factors of $\exp(-Rt_1)$ are canceled by the normalization. Equation (8.5.7) (and its generalizations) is the main result of this section. In particular, if we consider $\rho(0)$ to be a pure case density matrix $\rho(0) = |\psi(0)\rangle\langle\psi(0)|$, then Eq. (8.5.7) may be written as

$$\rho^{(n)}(t) = \frac{e^{-Ra^\dagger at}a^n|\psi(0)\rangle}{\sqrt{\langle\psi(0)|a^{\dagger n}e^{-2Ra^\dagger at}a^n|\psi(0)\rangle}} \frac{\langle\psi(0)|a^{\dagger n}e^{-Ra^\dagger at}}{\sqrt{\langle\psi(0)|a^{\dagger n}e^{-2Ra^\dagger at}a^n|\psi(0)\rangle}}. \tag{8.5.8}$$

Equation (8.5.8) provides a natural introduction to the wave function approach to dissipative processes.

8.5.2 The wave function Monte Carlo approach to damping

Motivated by the result of the previous section, i.e., Eq. (8.5.8), we present here a short account of damping via a wave function approach. In order to present the ideas we will continue to consider the simple problem of a damped single-mode field, but we will have a more general reservoir, such as that in Section 8.3, in mind. Thus, the decay rate R is no longer governed by the time τ but by the much shorter reservoir correlation times. From Eq. (8.5.8) we are led to write the 'conditional state vector'

$$|\psi^{(n)}(t+\delta t)\rangle = \frac{e^{-Ra^\dagger a\delta t}a^n|\psi(t)\rangle}{\sqrt{\langle\psi(t)|a^{\dagger n}e^{-2Ra^\dagger a\delta t}a^n|\psi(t)\rangle}}, \tag{8.5.9}$$

which represents the state of the field under the condition of n photons absorbed in time δt starting from $|\psi(t)\rangle$. In particular, the state involving only zero or one such event is of special interest. That is, the state at time $t + \delta t$ for $n = 0$ (a null measurement) is

$$\begin{aligned} |\psi^{(0)}(t+\delta t)\rangle &= \frac{e^{-Ra^\dagger a\delta t}|\psi(t)\rangle}{\sqrt{\langle\psi(t)|e^{-2Ra^\dagger a\delta t}|\psi(t)\rangle}} \\ &\cong \frac{(1 - Ra^\dagger a\delta t)}{\sqrt{1 - 2R\langle a^\dagger a\rangle\delta t}}|\psi(t)\rangle, \end{aligned} \tag{8.5.10a}$$

where $\langle a^\dagger a\rangle = \langle\psi(t)|a^\dagger a|\psi(t)\rangle$; and the state corresponding to $n = 1$ (quantum jump) is

$$\begin{aligned} |\psi^{(1)}(t+\delta t)\rangle &= \frac{e^{-Ra^\dagger a\delta t}a|\psi(t)\rangle}{\sqrt{\langle\psi(t)|a^\dagger e^{-2Ra^\dagger a\delta t}a|\psi(t)\rangle}} \\ &\cong \frac{a}{\sqrt{\langle a^\dagger a\rangle}}|\psi(t)\rangle. \end{aligned} \tag{8.5.10b}$$

For example, if the initial quantum state for the field mode is

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle, \tag{8.5.11}$$

then (8.5.10a) and (8.5.10b) imply the conditional state vectors

$$|\psi^{(0)}(t+\delta t)\rangle = \frac{c_0|0\rangle + c_1(1-R\delta t)|1\rangle}{\sqrt{1-2c_1^*c_1R\delta t}}, \tag{8.5.12a}$$

and

$$|\psi^{(1)}(t+\delta t)\rangle = \frac{c_1}{\sqrt{|c_1|^2}}|0\rangle. \tag{8.5.12b}$$

However, we want to describe the evolution from $|\psi(t)\rangle$ as given by Eq. (8.5.11) to a general state at later times which must become $|0\rangle$ eventually. During the time δt the unnormalized 'no count' or ' null measurement' $|\tilde{\psi}\rangle$ is seen from Eq. (8.5.10a) to obey the equation of motion

$$\frac{|\tilde{\psi}^{(0)}(t+\delta t)\rangle - |\tilde{\psi}^{(0)}(t)\rangle}{\delta t} = -Ra^\dagger a|\tilde{\psi}^{(0)}(t)\rangle, \tag{8.5.13a}$$

that is

$$\frac{d}{dt}|\tilde{\psi}^{(0)}(t)\rangle = -\frac{i}{\hbar}(-i\hbar Ra^\dagger a)|\tilde{\psi}^{(0)}(t)\rangle. \tag{8.5.13b}$$

Thus we are motivated to describe the time evolution of the unnormalized state vector for the case of no absorption by a nonunitary Schrödinger equation

$$\frac{d}{dt}|\tilde{\psi}^{(0)}(t)\rangle = -\frac{i}{\hbar}\mathscr{V}_1|\tilde{\psi}^{(0)}(t)\rangle \tag{8.5.14}$$

governed by the non-Hermitian Hamiltonian

$$\mathscr{V}_1 = -i\hbar Ra^\dagger a. \tag{8.5.15}$$

The temporal development implied by Eq. (8.5.14) is, of course, interrupted by quantum jumps or collapses of the wave function at random times. When such a collapse occurs, the state is given by $|0\rangle$. This happens only once, from that time on the field is in the vacuum state. Continuing with our simple example, according to Eq. (8.5.14) the unnormalized state vector

$$|\tilde{\psi}(t)\rangle = \tilde{c}_0(t)|0\rangle + \tilde{c}_1(t)|1\rangle \tag{8.5.16}$$

obeys the simple equations of motion

$$\dot{\tilde{c}}_0(t) = 0, \tag{8.5.17a}$$

$$\dot{\tilde{c}}_1(t) = -R\tilde{c}_1(t), \tag{8.5.17b}$$

which imply

$$\tilde{c}_0(t) = \tilde{c}_0(0), \tag{8.5.18a}$$

$$\tilde{c}_1(t) = \tilde{c}_1(0)e^{-Rt}, \tag{8.5.18b}$$

and the corresponding normalized probability amplitudes

$$c_0(t) = \frac{c_0(0)}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-2Rt}}}, \tag{8.5.19a}$$

and

$$c_1(t) = \frac{c_1(0)e^{-Rt}}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-2Rt}}}. \tag{8.5.19b}$$

Thus we have the complete coherent evolution for the conditional state vector up to the point of collapse,

$$|\psi^{(0)}(t)\rangle = \frac{c_0(0)|0\rangle + c_1(0)e^{-Rt}|1\rangle}{\sqrt{|c_0(0)|^2 + |c_1(0)|^2 e^{-2Rt}}}. \tag{8.5.20}$$

Note that as $t \to \infty$ the state $|\psi^{(0)}(t)\rangle \to |0\rangle$. This is as it should be since the conditional state $|\psi^{(0)}(t)\rangle$ is that state which is conditioned on the premise that no photons are absorbed. Hence if after a long time we never see a 'count', then the conclusion is that we must have been in the vacuum state, $|0\rangle$, all along. To summarize: the field develops from $t = 0$ up to some time t according to Eq. (8.5.14), and between t and $t + \delta t$ a jump occurs, that is

$$|\psi(0)\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle \tag{8.5.21a}$$

$$\Big\downarrow \text{'no counts' from } 0 \to t$$

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle \tag{8.5.21b}$$

$$\Big\downarrow \text{collapse } t \to t + \delta t$$

$$\begin{aligned} |\psi(t + \delta t)\rangle &= \frac{a}{\sqrt{\langle\psi(t)|a^\dagger a|\psi(t)\rangle}}|\psi(t)\rangle \\ &= |0\rangle, \end{aligned} \tag{8.5.21c}$$

where $c_0(t)$ and $c_1(t)$ in (8.5.21b) are given by Eqs. (8.5.19a) and (8.5.19b) and Eq. (8.5.21c) follows from Eq. (8.5.10b). Now we recall that the probability of a collapse or jump at time t is governed by the density matrix conditional upon a single photon absorption, i.e., a 'count'. With that in mind, we write Eq. (8.3.3) for $R = \mathscr{C}/2$ as

$$\begin{aligned} \dot{\rho} &= -R\left(a^\dagger a\rho + \rho a^\dagger a\right) + 2Ra\rho a^\dagger \\ &= \underbrace{-\frac{i}{\hbar}\left(\mathscr{V}_1\rho - \rho\mathscr{V}_1^\dagger\right)} + \underbrace{2Ra\rho a^\dagger} \\ &= \quad \dot{\rho}(\text{no count}) \quad + \quad \dot{\rho}(\text{count}). \end{aligned} \tag{8.5.22}$$

Hence the probability for a collapse between t and $t + \delta t$ is given by

$$\begin{aligned}\mathrm{Tr}[\dot{\rho}(\mathrm{count})]\delta t &= 2R\delta t \mathrm{Tr}[\rho(t)a^\dagger a] \\ &= 2R\delta t\langle\psi(t)|a^\dagger a|\psi(t)\rangle \\ &= 2R\delta t\frac{\langle\tilde{\psi}(t)|a^\dagger a|\tilde{\psi}(t)\rangle}{\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle}. \end{aligned} \tag{8.5.23}$$

Therefore, from Eqs. (8.5.16)–(8.5.19b) and (8.5.23), we have the jump probability for our present problem

$$P_{\mathrm{jump}}(t) = 2R\delta t\frac{|c_1(0)|^2 e^{-2Rt}}{|c_0(0)|^2 + |c_1(0)|^2 e^{-2Rt}}. \tag{8.5.24}$$

Finally we turn the above into a plot of the probability of finding a photon in the cavity after a time t given that $c_0(0) = 0$ and $c_1(0) = 1$. Then $P_{\mathrm{jump}}(t) = 2R\delta t$. This we do via a Monte Carlo procedure as follows. First, we start the field in state $|1\rangle$ with $c_1(0) = 1$ and we choose a number between 0 and 1 using a computer random number generator. If the number is smaller than $P_{\mathrm{jump}}(0)$, then a jump or collapse is taken to have occurred, and the photon number is set to zero. Most likely, however, the number will be larger than P_{jump} and we reevaluate $|\psi(t)\rangle$ from (8.5.20) and start again. We repeat this n times until a random number turns up which is smaller than $P_{\mathrm{jump}}(t)$ given by (8.5.24). At that point we make an entry in our table as follows:

$$\begin{array}{ll}
t=0 & |\psi(0)\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle \\
 & \Big\downarrow \text{evolve} \\
t=\delta t & |\psi(\delta t)\rangle = c_0(\delta t)|0\rangle + c_1(\delta t)|1\rangle \\
 & \Big\downarrow \text{evolve} \\
t=2\delta t & |\psi(2\delta t)\rangle = c_0(2\delta t)|0\rangle + c_1(2\delta t)|1\rangle \\
 & \vdots \\
 & \Big\downarrow \text{evolve} \\
t=n\delta t & |\psi(n\delta t)\rangle = c_0(n\delta t)|0\rangle + c_1(n\delta t)|1\rangle \\
 & \Big\downarrow \text{collapse} \\
t=(n+1)\delta t & |\psi[(n+1)\delta t]\rangle = |0\rangle.
\end{array} \tag{8.5.25}$$

Needless to say, the preceding simple example was chosen for pedagogical purposes. Many more involved problems can be and have been solved by the quantum jump–Monte Carlo approach. These include spontaneous emission, resonance fluorescence, Doppler cooling, population trapping, and the dark line resonance, to name a few.

In conclusion we note that the approach of the present section is often referred to as the 'quantum trajectory method'. We also point to the interesting work of Willis Lamb in which the trajectories of Gaussian wave packets are calculated in order to treat the quantum theory of certain problems dealing with the measurement process. This work also uses a computer analysis to characterize the (random) outcomes of the experiment.

Problems

8.1 Derive Eqs. (8.2.14a) and (8.2.14b) and use these results to evaluate the correlation functions (8.2.16a)–(8.2.16e).

8.2 The equation of motion for the reduced density operator for a single-mode cavity field coupled to a vacuum reservoir through a partially transmitting mirror is

$$\dot{\rho} = -\frac{\mathscr{C}}{2}(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a).$$

Here $\mathscr{C}$ is the loss rate related to the Q-factor of the cavity by $\mathscr{C} = \nu/Q$. Derive the equations of motion for the relevant quantities, and then solve them to show that the variances $(\Delta X_1)_t^2$ and $(\Delta X_2)_t^2$ (with $X_1 = (a+a^\dagger)/2$ and $X_2 = (a-a^\dagger)/2i$) increase due to dissipation (fluctuation–dissipation theorem!). This situation can be viewed as a bosonic mode, uncorrelated to the cavity field, entering the cavity through the partially transmitting mirror, and hence adding the uncorrelated noise.

8.3 If the reservoir in the above problem is in a multi-mode squeezed vacuum state, the resulting equation of motion for the reduced density matrix is given by Eq. (8.3.4). As before, calculate the variances $(\Delta X_1)_t^2$ and $(\Delta X_2)_t^2$. Is it possible to suppress the added noise in this situation?

8.4 For a thermal reservoir

$$\dot{\rho} = -\frac{\mathscr{C}}{2}(\bar{n}_{\text{th}} + 1)(a^\dagger a\rho - 2a\rho a^\dagger + \rho a^\dagger a)$$
$$-\frac{\mathscr{C}}{2}\bar{n}_{\text{th}}(aa^\dagger\rho - 2a^\dagger\rho a + \rho aa^\dagger),$$

where $\bar{n}_{\text{th}}$ is the mean number of photons in the reservoir. Derive the corresponding equation for the Q-representation and solve it.

8.5 Derive Eqs. (8.2.18a)–(8.2.18c).

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