CHAPTER 6

Atom–field interaction – quantum theory

In the preceding chapters concerning the interaction of a radiation field with matter, we assumed the field to be classical. In many situations this assumption is valid. There are, however, many instances where a classical field fails to explain experimentally observed results and a quantized description of the field is required. This is, for example, true of spontaneous emission in an atomic system which was described phenomenologically in Chapter 5. For a rigorous treatment of the atomic level decay in free space, we need to consider the interaction of the atom with the vacuum modes of the universe. Even in the simplest system involving the interaction of a single-mode radiation field with a single two-level atom, the predictions for the dynamics of the atom are quite different in the semiclassical theory and the fully quantum theory. In the absence of the decay process, the semiclassical theory predicts Rabi oscillations for the atomic inversion whereas the quantum theory predicts certain *collapse* and *revival* phenomena due to the quantum aspects of the field. These interesting quantum field theoretical predictions have been experimentally verified.

In this chapter we discuss the interaction of the quantized radiation field with the two-level atomic system described by a Hamiltonian in the dipole and the rotating-wave approximations. For a single-mode field it reduces to a particularly simple form. This is a very interesting Hamiltonian in quantum optics for several reasons. First, it can be solved exactly for arbitrary coupling constants and exhibits some true quantum dynamical effects such as collapse followed by periodic revivals of the atomic inversion. Second, it provides the simplest illustration of spontaneous emission and thus explains the effects of various kinds of quantum statistics of the field in more complicated systems such as a micromaser and a laser, which we shall study in later chapters. Third, and perhaps most importantly,* it has become possible to realize it experimentally through the spectacular advances in the development of high-Q microwave cavities.

The spontaneous decay of an atomic level is treated by considering the interaction of the two-level atom with the modes of the universe in the vacuum state. We examine the state of the field that is generated in the process of emission of a quantum of energy equal to the energy difference between the atomic levels. Such a state may be regarded as a single-photon state.

6.1 Atom–field interaction Hamiltonian

The interaction of a radiation field **E** with a single-electron atom can be described by the following Hamiltonian in the dipole approximation:

$$\mathscr{H} = \mathscr{H}_A + \mathscr{H}_F - e\mathbf{r}\cdot\mathbf{E}. \tag{6.1.1}$$

Here $\mathscr{H}_A$ and $\mathscr{H}_F$ are the energies of the atom and the radiation field, respectively, in the absence of the interaction, and $\mathbf{r}$ is the position vector of the electron. In the dipole approximation, the field is assumed to be uniform over the whole atom.

The energy of the free field $\mathscr{H}_F$ is given in terms of the creation and destruction operators by

$$\mathscr{H}_F = \sum_{\mathbf{k}} \hbar\nu_k \left(a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2}\right). \tag{6.1.2}$$

We can express $\mathscr{H}_A$ and $e\mathbf{r}$ in terms of the atom transition operators

$$\sigma_{ij} = |i\rangle\langle j|. \tag{6.1.3}$$

As before $\{|i\rangle\}$ represents a complete set of atomic energy eigenstates, i.e., $\sum_i |i\rangle\langle i| = 1$. It then follows from the eigenvalue equation $\mathscr{H}_A|i\rangle = E_i|i\rangle$ that

$$\mathscr{H}_A = \sum_i E_i|i\rangle\langle i| = \sum_i E_i\sigma_{ii}. \tag{6.1.4}$$

Also

$$e\mathbf{r} = \sum_{i,j} e|i\rangle\langle i|\mathbf{r}|j\rangle\langle j| = \sum_{i,j} \wp_{ij}\sigma_{ij}, \tag{6.1.5}$$

* Especially the micromaser of H. Walther and coworkers as discussed in Chapter 13. See also the Physics Today article by Haroche and Kleppner [1989] which presents the physics of cavity QED very nicely.

where $\wp_{ij} = e\langle i|\mathbf{r}|j\rangle$ is the electric–dipole transition matrix element. The electric field operator is evaluated in the dipole approximation at the position of the point atom. It follows from Eq. (1.1.27) that, for the atom at the origin, we have

$$\mathbf{E} = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} (a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger}), \tag{6.1.6}$$

where $\mathscr{E}_{\mathbf{k}} = (\hbar \nu_k / 2\epsilon_0 V)^{1/2}$. Here, for simplicity, we have taken a linear polarization basis and the polarization unit vectors to be real.

It now follows, on substituting for $\mathscr{H}_F, \mathscr{H}_A, e\mathbf{r}$, and $\mathbf{E}$ from Eqs. (6.1.2), (6.1.4), (6.1.5), and (6.1.6) into Eq. (6.1.1), that

$$\mathscr{H} = \sum_{\mathbf{k}} \hbar \nu_k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \sum_i E_i \sigma_{ii} + \hbar \sum_{i,j} \sum_{\mathbf{k}} g_{\mathbf{k}}^{ij} \sigma_{ij} (a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger}), \tag{6.1.7}$$

where

$$g_{\mathbf{k}}^{ij} = -\frac{\wp_{ij} \cdot \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}}}{\hbar}. \tag{6.1.8}$$

In Eq. (6.1.7), we have omitted the zero-point energy from the first term. For the sake of simplicity, we will assume $\wp_{ij}$ to be real throughout this chapter.

We now proceed to the case of a two-level atom. For $\wp_{ab} = \wp_{ba}$, we write

$$g_{\mathbf{k}} = g_{\mathbf{k}}^{ab} = g_{\mathbf{k}}^{ba}. \tag{6.1.9}$$

The following form of the Hamiltonian is obtained

$$\begin{aligned} \mathscr{H} = &\sum_{\mathbf{k}} \hbar \nu_k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + (E_a \sigma_{aa} + E_b \sigma_{bb}) \\ &+ \hbar \sum_{\mathbf{k}} g_{\mathbf{k}} (\sigma_{ab} + \sigma_{ba})(a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger}). \end{aligned} \tag{6.1.10}$$

The second term in Eq. (6.1.10) can be rewritten as

$$E_a \sigma_{aa} + E_b \sigma_{bb} = \frac{1}{2} \hbar \omega (\sigma_{aa} - \sigma_{bb}) + \frac{1}{2}(E_a + E_b), \tag{6.1.11}$$

where we use $(E_a - E_b) = \hbar\omega$ and $\sigma_{aa} + \sigma_{bb} = 1$. The constant energy term $(E_a + E_b)/2$ can be ignored. If we use the notation

$$\sigma_z = \sigma_{aa} - \sigma_{bb} = |a\rangle\langle a| - |b\rangle\langle b|, \tag{6.1.12}$$

$$\sigma_+ = \sigma_{ab} = |a\rangle\langle b|, \tag{6.1.13}$$

$$\sigma_- = \sigma_{ba} = |b\rangle\langle a|, \tag{6.1.14}$$

the Hamiltonian (6.1.10) takes the form

$$\mathscr{H}=\sum_{\mathbf{k}}\hbar\nu_k a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+\frac{1}{2}\hbar\omega\sigma_z+\hbar\sum_{\mathbf{k}}g_{\mathbf{k}}(\sigma_{+}+\sigma_{-})(a_{\mathbf{k}}+a_{\mathbf{k}}^{\dagger}). \quad (6.1.15)$$

It follows from the identity

$$[\sigma_{ij},\sigma_{kl}]=\sigma_{il}\delta_{jk}-\sigma_{kj}\delta_{il}, \quad (6.1.16)$$

that σ_{+},σ_{-}, and σ_z satisfy the spin-1/2 algebra of the Pauli matrices, i.e.,

$$[\sigma_{-},\sigma_{+}]=-\sigma_z, \quad (6.1.17)$$

$$[\sigma_{-},\sigma_z]=2\sigma_{-}. \quad (6.1.18)$$

In the matrix notation, σ_{-},σ_{+}, and σ_z are given by

$$\sigma_{-}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \sigma_{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \quad (6.1.19)$$

The σ_{-} operator takes an atom in the upper state into the lower state whereas σ_{+} takes an atom in the lower state into the upper state.

The interaction energy in Eq. (6.1.15) consists of four terms. The term $a_{\mathbf{k}}^{\dagger}\sigma_{-}$ describes the process in which the atom is taken from the upper state into the lower state and a photon of mode **k** is created. The term $a_{\mathbf{k}}\sigma_{+}$ describes the opposite process. The energy is conserved in both the processes. The term $a_{\mathbf{k}}\sigma_{-}$ describes the process in which the atom makes a transition from the upper to the lower level and a photon is annihilated, resulting in the loss of approximately $2\hbar\omega$ in energy. Similarly $a_{\mathbf{k}}^{\dagger}\sigma_{+}$ results in the gain of $2\hbar\omega$. Dropping the energy nonconserving terms corresponds to the rotating-wave approximation. The resulting simplified Hamiltonian is

$$\mathscr{H}=\sum_{\mathbf{k}}\hbar\nu_k a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+\frac{1}{2}\hbar\omega\sigma_z+\hbar\sum_{\mathbf{k}}g_{\mathbf{k}}(\sigma_{+}a_{\mathbf{k}}+a_{\mathbf{k}}^{\dagger}\sigma_{-}). \quad (6.1.20)$$

This form of the Hamiltonian describing the interaction of a single two-level atom with a multi-mode field is the starting point of many calculations in the field of quantum optics.

6.2 Interaction of a single two-level atom with a single-mode field

It follows from Eq. (6.1.20) that the interaction of a single-mode quantized field of frequency ν with a single two-level atom is described by the Hamiltonian

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \tag{6.2.1}$$

where

$$\mathscr{H}_0 = \hbar \nu a^\dagger a + \frac{1}{2} \hbar \omega \sigma_z, \tag{6.2.2}$$

$$\mathscr{H}_1 = \hbar g(\sigma_+ a + a^\dagger \sigma_-). \tag{6.2.3}$$

Here we have removed the subscript from the coupling constant g. The Hamiltonian, given by Eqs. (6.2.1)–(6.2.3), describes the atom–field interaction in the dipole and rotating-wave approximations. As we show below, this important Hamiltonian of quantum optics provides us with an exactly solvable example of the field–matter interaction.

It is convenient to work in the interaction picture. The Hamiltonian, in the interaction picture, is given by

$$\mathscr{V} = e^{i\mathscr{H}_0 t/\hbar} \mathscr{H}_1 e^{-i\mathscr{H}_0 t/\hbar}. \tag{6.2.4}$$

Using

$$e^{\alpha A} B e^{-\alpha A} = B + \alpha[A, B] + \frac{\alpha^2}{2!}[A, [A, B]] + \ldots, \tag{6.2.5}$$

it can be readily seen that

$$e^{i\nu a^\dagger a t} a e^{-i\nu a^\dagger a t} = a e^{-i\nu t}, \tag{6.2.6}$$

$$e^{i\omega\sigma_z t/2} \sigma_+ e^{-i\omega\sigma_z t/2} = \sigma_+ e^{i\omega t}. \tag{6.2.7}$$

Combining Eqs. (6.2.1)–(6.2.3), (6.2.4), (6.2.6), and (6.2.7), we have

$$\mathscr{V} = \hbar g(\sigma_+ a e^{i\Delta t} + a^\dagger \sigma_- e^{-i\Delta t}), \tag{6.2.8}$$

where $\Delta = \omega - \nu$.

In this section, we present three different but equivalent methods to solve for the evolution of the atom–field system described by the Hamiltonian (6.2.1)–(6.2.3) based on the solutions of the probability amplitudes, the Heisenberg field and atomic operators, and the unitary time-evolution operator.

6.2.1 Probability amplitude method

We first proceed to solve the equation of motion for $|\psi\rangle$, i.e.,

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \mathscr{V}|\psi\rangle. \tag{6.2.9}$$

At any time t, the state vector $|\psi(t)\rangle$ is a linear combination of the states $|a, n\rangle$ and $|b, n\rangle$. Here $|a, n\rangle$ is the state in which the atom is in the excited state $|a\rangle$ and the field has n photons. A similar description exists for the state $|b, n\rangle$. As we are using the interaction picture, we use the slowly varying probability amplitudes $c_{a,n}$ and $c_{b,n}$. The state vector is therefore

$$|\psi(t)\rangle = \sum_n [c_{a,n}(t)|a, n\rangle + c_{b,n}(t)|b, n\rangle]. \tag{6.2.10}$$

The interaction energy (6.2.8) can only cause transitions between the states $|a, n\rangle$ and $|b, n+1\rangle$. We therefore consider the evolution of the amplitudes $c_{a,n}$ and $c_{b,n+1}$. The equations of motion for the probability amplitudes $c_{a,n}$ and $c_{b,n+1}$ are obtained by first substituting for $|\psi(t)\rangle$ and $\mathscr{V}$ from Eqs. (6.2.10) and (6.2.8) in Eq. (6.2.9) and then projecting the resulting equations onto $\langle a, n|$ and $\langle b, n+1|$, respectively. We then obtain

$$\dot{c}_{a,n} = -ig\sqrt{n+1}\, e^{i\Delta t} c_{b,n+1}, \tag{6.2.11}$$

$$\dot{c}_{b,n+1} = -ig\sqrt{n+1}\, e^{-i\Delta t} c_{a,n}. \tag{6.2.12}$$

This coupled set of equations is very similar to that obtained in the semiclassical treatment (cf. Eqs. (5.2.12) and (5.2.13)). These equations can be solved exactly subject to certain initial conditions. A general solution for the probability amplitudes is

$$c_{a,n}(t) = \left\{ c_{a,n}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} c_{b,n+1}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{i\Delta t/2}, \tag{6.2.13}$$

$$c_{b,n+1}(t) = \left\{ c_{b,n+1}(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) + \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] - \frac{2ig\sqrt{n+1}}{\Omega_n} c_{a,n}(0) \sin\left(\frac{\Omega_n t}{2}\right) \right\} e^{-i\Delta t/2}, \tag{6.2.14}$$

where

$$\Omega_n^2 = \Delta^2 + 4g^2(n+1). \tag{6.2.15}$$

If initially the atom is in the excited state $|a\rangle$ then $c_{a,n}(0) = c_n(0)$ and $c_{b,n+1}(0) = 0$. Here $c_n(0)$ is the probability amplitude for the field alone. We then obtain

$$c_{a,n}(t) = c_n(0) \left[\cos\left(\frac{\Omega_n t}{2}\right) - \frac{i\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) \right] e^{i\Delta t/2}, \tag{6.2.16}$$

$$c_{b,n+1}(t) = -c_n(0) \frac{2ig\sqrt{n+1}}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) e^{-i\Delta t/2}. \tag{6.2.17}$$

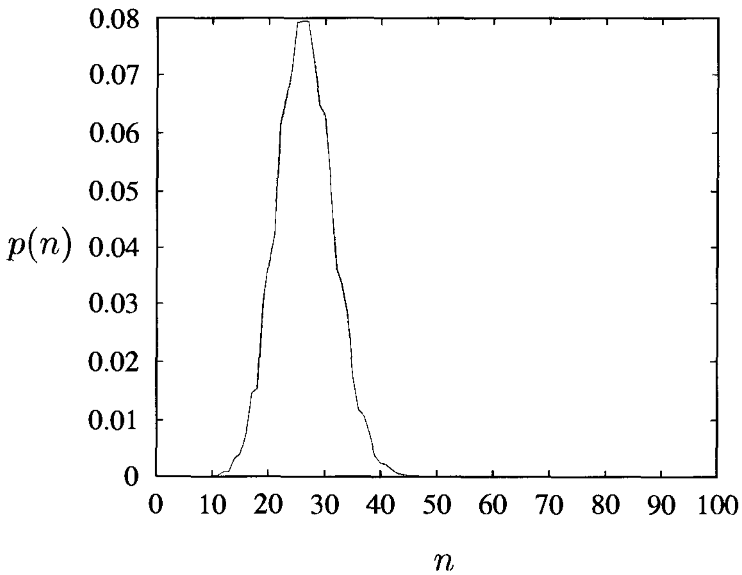


Fig. 6.1
Behavior of $p(n)$, as given by Eq. (6.2.18), for an initially coherent state. The value of the various parameters are $\Delta = 0$, $\langle n \rangle = 25$, and $gt = 1$.

These equations give us a complete solution of the problem. All the physically relevant quantities relating to the quantized field and the atom can be obtained from them.

The expressions $|c_{a,n}(t)|^2$ and $|c_{b,n}(t)|^2$ represent the probabilities that, at time t, the field has n photons present and the atom is in levels $|a\rangle$ and $|b\rangle$, respectively. The probability $p(n)$ that there are n photons in the field at time t is therefore obtained by taking the trace over the atomic states, i.e.,

$$p(n) = |c_{a,n}(t)|^2 + |c_{b,n}(t)|^2$$
$$= \rho_{nn}(0)\left[\cos^2\left(\frac{\Omega_n t}{2}\right) + \left(\frac{\Delta}{\Omega_n}\right)^2 \sin^2\left(\frac{\Omega_n t}{2}\right)\right]$$
$$+\rho_{n-1,n-1}(0)\left(\frac{4g^2 n}{\Omega_{n-1}^2}\right)\sin^2\left(\frac{\Omega_{n-1} t}{2}\right), \tag{6.2.18}$$

where $\rho_{nn}(0) = |c_n(0)|^2$ is the probability that there are n photons present in the field at time $t = 0$. In Fig. 6.1, we plot $p(n)$ for an initial coherent state

$$\rho_{nn}(0) = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}. \tag{6.2.19}$$

Another important quantity is the inversion $W(t)$ which is related to the probability amplitudes $c_{a,n}(t)$ and $c_{b,n}(t)$ by the expression

$$W(t) = \sum_n \left[|c_{a,n}(t)|^2 - |c_{b,n}(t)|^2\right]. \tag{6.2.20}$$

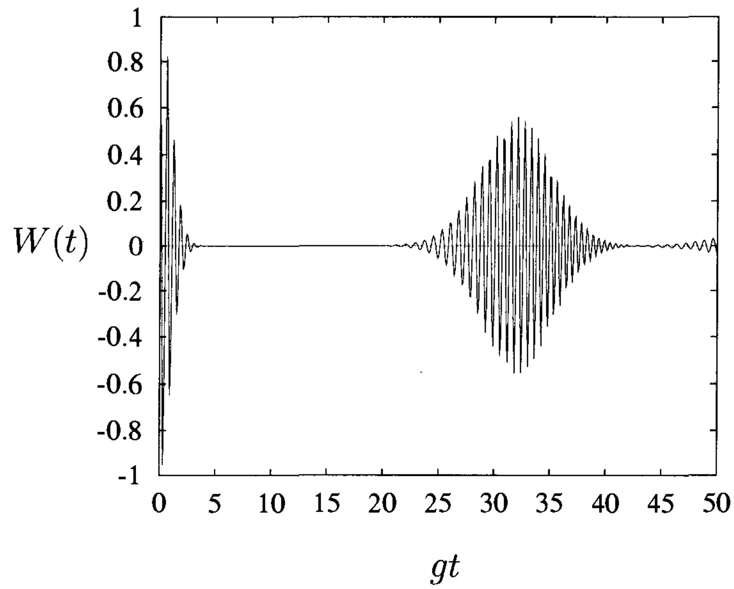


Fig. 6.2
Time evolution of the population inversion $W(t)$ for an initially coherent state with $\langle n \rangle = 25$ and $\Delta = 0$.

On substituting for $c_{a,n}(t)$ and $c_{b,n}(t)$ from Eqs. (6.2.16) and (6.2.17) and making some rearrangements, we obtain

$$W(t) = \sum_{n=0}^{\infty} \rho_{nn}(0) \left[\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2} \cos(\Omega_n t) \right]. \tag{6.2.21}$$

It is interesting to note that even for initial vacuum field ($\rho_{nn}(0) = \delta_{n0}$),

$$W(t) = \frac{1}{\Delta^2 + 4g^2} \left\{ \Delta^2 + 4g^2 \cos\left[\left(\Delta^2 + 4g^2\right)^{1/2} t \right] \right\}, \tag{6.2.22}$$

i.e., the Rabi oscillations take place. This result is drastically different from the predictions of the semiclassical theory of Chapter 5. In the semiclassical theory, the atom in the excited state cannot make a transition to the lower level in the absence of a driving field. In the fully quantum mechanical treatment, the transition from the upper level to the lower level in the *vacuum* becomes possible due to spontaneous emission. Equation (6.2.22) is the simplest example of spontaneous emission in which the spontaneously emitted photon contributes to the single mode of the field considered. A detailed analysis of spontaneous emission by an atom in free space due to the presence of infinitely many vacuum modes will be discussed in the next section.

In Fig. 6.2, $W(t)$ is plotted as a function of the normalized time $\tau = gt$ for an initial coherent state. The behavior of $W(t)$ is quite different from the corresponding curve (Fig. 5.2) in the semiclassical theory. In the present case the envelope of the sinusoidal Rabi oscillations 'collapses' to zero. However as time increases we encounter a 'revival' of the collapsed inversion. This behavior of collapse and revival of inversion is repeated with increasing time, with the amplitude of Rabi oscillations decreasing and the time duration in which revival takes place increasing and ultimately overlapping with the earlier revival.

The phenomena of collapse and revival can be physically understood from Eq. (6.2.21). Each term in the summation represents Rabi oscillations for a definite value of n. The photon distribution function $\rho_{nn}(0)$ determines the relative weight for each value of n. At the initial time, $t = 0$, the atom is prepared in a definite state and therefore all the terms in the summation are correlated. As time increases the Rabi oscillations associated with different excitations have different frequencies and therefore become uncorrelated leading to a collapse of inversion. As time is further increased, the correlation is restored and revival occurs. This behavior continues and an infinite sequence of revivals is obtained. The important thing is that revivals occur only because of the granular structure of the photon distribution. Revival is thus a pure quantum phenomenon. A continuous photon distribution (without zeros) would give a collapse, as would a classical random field, but no revivals.

Simple expressions for the times t_R, t_c, and t_r associated with the sinusoidal Rabi oscillations, the collapse of these oscillations and their revival, respectively, can be determined from Eq. (6.2.21) in the limit $\langle n\rangle \gg 1$. The time period t_R of the Rabi oscillations is given by the inverse of the Rabi frequency Ω_n at $n = \langle n\rangle$, i.e.,

$$t_R \sim \frac{1}{\Omega_{\langle n\rangle}} = \frac{1}{(\Delta^2 + 4g^2\langle n\rangle)^{1/2}}. \tag{6.2.23}$$

As mentioned earlier, these Rabi oscillations continue until a collapse time t_c, when the oscillations associated with different values of n become uncorrelated. Now, for the Poisson distribution (6.2.19) for the initial coherent field, the root-mean-square deviation in the photon number Δn is equal to $\sqrt{\langle n\rangle}$. An estimate of t_c can therefore be obtained from the condition

$$\left(\Omega_{\langle n\rangle+\sqrt{\langle n\rangle}} - \Omega_{\langle n\rangle-\sqrt{\langle n\rangle}}\right) t_c \sim 1. \tag{6.2.24}$$

Since $\langle n\rangle \gg \sqrt{\langle n\rangle}$ in the limit $\langle n\rangle \gg 1$, Eq. (6.2.24) yields

$$\begin{aligned} t_c &\sim \frac{1}{\Omega_{\langle n\rangle+\sqrt{\langle n\rangle}} - \Omega_{\langle n\rangle-\sqrt{\langle n\rangle}}} \\ &\simeq \frac{1}{\left[\Delta^2+4g^2\left(\langle n\rangle+\sqrt{\langle n\rangle}\right)\right]^{1/2}-\left[\Delta^2+4g^2\left(\langle n\rangle-\sqrt{\langle n\rangle}\right)\right]^{1/2}} \\ &\simeq \frac{1}{2g}\left(1+\frac{\Delta^2}{4g^2\langle n\rangle}\right)^{1/2}. \end{aligned} \tag{6.2.25}$$

Under the conditions of exact resonance, $\Delta = 0$, the collapse time t_c is equal to $1/2g$ and is independent of the mean number of photons $\langle n\rangle$. For nonzero detuning, t_c decreases with increasing $\langle n\rangle$. The interval between revivals, t_r, can be found from the condition

$$(\Omega_{\langle n\rangle} - \Omega_{\langle n\rangle-1})t_r = 2\pi m \quad (m = 1, 2, \ldots), \tag{6.2.26}$$

i.e., the revivals take place when the phases of oscillation of the neighboring terms in Eq. (6.2.21) differ by an integral multiple of 2π. Again, in the limit $\langle n\rangle \gg 1$, we obtain

$$\begin{aligned} t_r &= \frac{2\pi m}{\Omega_{\langle n\rangle} - \Omega_{\langle n\rangle-1}} \\ &\simeq \frac{2\pi m\sqrt{\langle n\rangle}}{g}\left(1+\frac{\Delta^2}{4g^2\langle n\rangle}\right)^{1/2}, \end{aligned} \tag{6.2.27}$$

where m is an integer. This shows that revivals take place at regular intervals.

6.2.2 Heisenberg operator method

So far we have considered the problem of the interaction of a single-mode quantized field with a single two-level atom in the interaction picture. In the following we give the solution of the same problem in the Heisenberg picture. In particular we solve the operator equations for the atomic and field operators $a(t)$ and $\sigma_\pm(t)$. These solutions may be particularly useful in the calculation of the multi-time correlation functions necessary in the study of the spectral properties of the field.

The Heisenberg equations for the operators a, σ_-, and σ_z are obtained from the atom–field Hamiltonian (6.2.1)

$$\dot{a} = \frac{1}{i\hbar}[a, \mathscr{H}] = -i\nu a - ig\sigma_-, \tag{6.2.28}$$

$$\dot{\sigma}_- = -i\omega\sigma_- + ig\sigma_z a, \tag{6.2.29}$$

$$\dot{\sigma}_z = 2ig(a^\dagger\sigma_- - \sigma_+ a). \tag{6.2.30}$$

In order to facilitate a solution of these coupled operator equations we define the following constants of motion:

$$N = a^\dagger a + \sigma_+\sigma_-, \tag{6.2.31}$$

$$C = \frac{1}{2}\Delta\sigma_z + g(\sigma_+ a + a^\dagger \sigma_-), \tag{6.2.32}$$

i.e., N and C commute with the Hamiltonian $[N, \mathscr{H}] = [C, \mathscr{H}] = 0$. Here N is an operator that represents the total excitation in the atom–field system, and C is an exchange constant.

We first derive an equation of motion for the atomic lowering operator σ_-. It follows from Eq. (6.2.29) that

$$\begin{aligned}\ddot{\sigma}_- &= -i\omega\dot{\sigma}_- + ig(\dot{\sigma}_z a + \sigma_z \dot{a}) \\ &= -i\omega\dot{\sigma}_- - 2g^2(a^\dagger\sigma_- a - \sigma_+ a^2) + \nu g\sigma_z a - g^2\sigma_-,\end{aligned} \tag{6.2.33}$$

where, in the second line, we substituted for $\dot{\sigma}_z$ and $\dot{a}$ from Eqs. (6.2.30) and (6.2.28), respectively. It is readily verified that

$$\begin{aligned} g^2(a^\dagger\sigma_- a - \sigma_+ a^2) = &-i\left(\frac{\Delta}{2} + C\right)\dot{\sigma}_- \\ &+ \left(\nu C - \frac{1}{2}\Delta^2 + \frac{1}{2}\omega\Delta\right)\sigma_-, \end{aligned} \tag{6.2.34}$$

$$g\sigma_z a = -i\dot{\sigma}_- + \omega\sigma_-. \tag{6.2.35}$$

On substituting these expressions in Eq. (6.2.33), we obtain the desired equation for σ_-:

$$\ddot{\sigma}_- + 2i(\nu - C)\dot{\sigma}_- + (2\nu C - \nu^2 + g^2)\sigma_- = 0. \tag{6.2.36}$$

In a similar manner, we obtain

$$\ddot{a} + 2i(\nu - C)\dot{a} + (2\nu C - \nu^2 + g^2)a = 0. \tag{6.2.37}$$

These equations can be solved in a straightforward manner and the resulting expressions for $\sigma_-(t)$ and $a(t)$ are

$$\begin{aligned}\sigma_-(t) &= [\sigma_+(t)]^\dagger \\ &= e^{-i\nu t}e^{iCt}\left[\left(\cos\kappa t + iC\frac{\sin\kappa t}{\kappa}\right)\sigma_-(0) - ig\frac{\sin\kappa t}{\kappa}a(0)\right],\end{aligned} \tag{6.2.38}$$

$$a(t) = e^{-i\nu t}e^{iCt}\left[\left(\cos\kappa t - iC\frac{\sin\kappa t}{\kappa}\right)a(0) - ig\frac{\sin\kappa t}{\kappa}\sigma_-(0)\right], \tag{6.2.39}$$

where κ is a constant operator

$$\kappa = \left[\frac{\Delta^2}{4} + g^2(N+1)\right]^{1/2}, \tag{6.2.40}$$

which commutes with C, i.e., $[C, \kappa] = 0$. In deriving Eqs. (6.2.38) and (6.2.39), we used

$$C^2 = \frac{\Delta^2}{4} + g^2 N, \tag{6.2.41}$$
$$g\sigma_z a = 2C\sigma_- + \Delta\sigma_- - ga. \tag{6.2.42}$$

Equations (6.2.38) and (6.2.39) provide a complete solution of the problem involving interaction of a two-level atom with a single-mode field in the Heisenberg picture. All quantities of interest can be obtained from these solutions. For example, the expression for the inversion $W(t)$ (Eq. (6.2.21)) can be recovered from Eq. (6.2.38) via

$$\begin{aligned} W(t) &= \langle a, \alpha | \sigma_z(t) | a, \alpha \rangle, \\ &= 2\langle a, \alpha | \sigma_+(t)\sigma_-(t) | a, \alpha \rangle - 1. \end{aligned} \tag{6.2.43}$$

Here we have assumed that the atom is intially in the excited state $|a\rangle$ and the field is intially in the coherent state $|\alpha\rangle$.

As mentioned earlier, a particular advantage of working in the Heisenberg picture is that the evaluation of multi-time correlation functions is straightforward. As an example, we can use Eq. (6.2.38) to construct the dipole–dipole correlation function (Problem 6.5)

$$\begin{aligned} &\langle a, \alpha | \sigma_+(t)\sigma_-(t+\tau) | a, \alpha \rangle \\ &= e^{-i\nu\tau - |\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\ &\times \frac{1}{4\Omega_n^2}\left[\cos(\Omega_{n-1}\tau/2) - \frac{i\Delta}{2\Omega_{n-1}}\sin(\Omega_{n-1}\tau/2)\right] \\ &\times \{(\Omega_n + \Delta)^2 e^{-i\Omega_n\tau/2} + (\Omega_n - \Delta)^2 e^{i\Omega_n\tau/2} \\ &+ 8g^2(n+1)\cos\left[\Omega_n(\tau + 2t)/2\right]\}, \end{aligned} \tag{6.2.44}$$

where Ω_n is given in Eq. (6.2.15).

6.2.3 Unitary time-evolution operator method

Another equivalent approach to deal with the problem of atom–field interaction is through the unitary time-evolution operator. In many problems where the evolution of the system is unitary, i.e., there is no dissipation, this approach may prove to be the simplest.

For the present problem of the interaction of a two-level atom with a single-mode quantized radiation field, the unitary time-evolution operation is given by

$$U(t) = \exp(-i\mathscr{V}t/\hbar), \tag{6.2.45}$$

where the interaction picture Hamiltonian $\mathscr{V}$, at exact resonance, is given by (Eq. (6.2.8) with $\Delta = 0$)

$$\mathscr{V} = \hbar g(\sigma_+ a + a^\dagger \sigma_-). \tag{6.2.46}$$

Here $\sigma_+ = |a\rangle\langle b|$ and $\sigma_- = |b\rangle\langle a|$. It follows, on using

$$(\sigma_+ a + a^\dagger \sigma_-)^{2\ell} = (aa^\dagger)^\ell |a\rangle\langle a| + (a^\dagger a)^\ell |b\rangle\langle b|, \tag{6.2.47}$$

$$(\sigma_+ a + a^\dagger \sigma_-)^{2\ell+1} = (aa^\dagger)^\ell a|a\rangle\langle b| + a^\dagger (aa^\dagger)^\ell |b\rangle\langle a|, \tag{6.2.48}$$

that

$$\begin{aligned} U(t) = {} & \cos(gt\sqrt{a^\dagger a + 1})|a\rangle\langle a| + \cos(gt\sqrt{a^\dagger a})|b\rangle\langle b| \\ & -i\frac{\sin(gt\sqrt{a^\dagger a + 1})}{\sqrt{a^\dagger a + 1}} a|a\rangle\langle b| - ia^\dagger \frac{\sin(gt\sqrt{a^\dagger a + 1})}{\sqrt{a^\dagger a + 1}} |b\rangle\langle a|. \end{aligned} \tag{6.2.49}$$

The wave vector at time t in terms of the wave vector at time $t = 0$ is simply given by

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle. \tag{6.2.50}$$

As an example of the equivalence of this method with earlier approaches we evaluate the probability amplitudes $c_{a,n}(t)$ and $c_{b,n+1}(t)$ for an atom initially in the excited state $|a\rangle$ and the field as a linear combination of number states, i.e.,

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n(0)|a, n\rangle. \tag{6.2.51}$$

On substituting for $U(t)$ and $|\psi(0)\rangle$ from Eqs. (6.2.49) and (6.2.51), respectively, in Eq. (6.2.50), we obtain

$$\begin{aligned} |\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(0) \Big[& \cos(gt\sqrt{n+1})|a, n\rangle \\ & - i\sin(gt\sqrt{n+1})|b, n+1\rangle \Big]. \end{aligned} \tag{6.2.52}$$

We thus have

$$c_{a,n}(t) = \langle a, n|\psi(t)\rangle = c_n(0)\cos(gt\sqrt{n+1}), \tag{6.2.53}$$

$$c_{b,n+1}(t) = -ic_n(0)\sin(gt\sqrt{n+1}), \tag{6.2.54}$$

in full agreement with Eqs. (6.2.16) and (6.2.17) for $\Delta = 0$.

6.3 Weisskopf–Wigner theory of spontaneous emission between two atomic levels

In the previous section, we showed that an atom in the upper level can make transitions back and forth to the lower state in time even in the absence of an applied field. However, it is seen experimentally that an atom in an excited state decays to the ground state with a characteristic lifetime but it does not make back and forth transitions. The atomic decay has been added into the atomic density matrix equations (see Problem 5.2) phenomenologically. In our model of spontaneous emission discussed in the previous section, the decay is not included because we have considered only one mode of the field. For a proper account of the atomic decay a continuum of modes, corresponding to a quantization cavity which is infinite in extent, needs to be included.

The interaction picture Hamiltonian, in the rotating-wave approximation, for this system is

$$\mathscr{V} = \hbar \sum_{\mathbf{k}} \left[g_{\mathbf{k}}^*(\mathbf{r}_0)\sigma_+ a_{\mathbf{k}} e^{i(\omega-\nu_k)t} + \text{H.c.} \right], \tag{6.3.1}$$

where $g_{\mathbf{k}}(\mathbf{r}_0) = g_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{r}_0)$, i.e., we have included the spatial dependence explicitly. Here, $\mathbf{r}_0$ is the location of the atom. The interaction picture Hamiltonian is obtained following the same method as outlined in the beginning of Section 6.2. We assume that at time $t = 0$ the atom is in the excited state $|a\rangle$ and the field modes are in the vacuum state $|0\rangle$. The state vector is therefore

$$|\psi(t)\rangle = c_a(t)|a, 0\rangle + \sum_{\mathbf{k}} c_{b,\mathbf{k}}|b, 1_{\mathbf{k}}\rangle, \tag{6.3.2}$$

with

$$c_a(0) = 1, \quad c_{b,\mathbf{k}}(0) = 0. \tag{6.3.3}$$

We want to determine the state of the atom and the state of the radiation field at some later time when the atom begins to emit photons and we do so in the Weisskopf–Wigner approximation.

From the Schrödinger equation

$$|\dot{\psi}(t)\rangle = -\frac{i}{\hbar}\mathscr{V}|\psi(t)\rangle, \tag{6.3.4}$$

we get the equations of motion for the probability amplitudes c_a and $c_{b,\mathbf{k}}$:

$$\dot{c}_a(t) = -i \sum_{\mathbf{k}} g_{\mathbf{k}}^*(\mathbf{r}_0) e^{i(\omega-\nu_k)t} c_{b,\mathbf{k}}(t), \tag{6.3.5}$$

$$\dot{c}_{b,\mathbf{k}}(t) = -i g_{\mathbf{k}}(\mathbf{r}_0) e^{-i(\omega-\nu_k)t} c_a(t). \tag{6.3.6}$$

In order to get an equation that involves c_a only, we first integrate Eq. (6.3.6),

$$c_{b,\mathbf{k}}(t) = -ig_{\mathbf{k}}(\mathbf{r}_0) \int_0^t dt' e^{-i(\omega-\nu_k)t'} c_a(t'). \tag{6.3.7}$$

On substituting this expression of $c_{b,\mathbf{k}}(t)$ into Eq. (6.3.5), we obtain

$$\dot{c}_a(t) = -\sum_{\mathbf{k}} |g_{\mathbf{k}}(\mathbf{r}_0)|^2 \int_0^t dt' e^{i(\omega-\nu_k)(t-t')} c_a(t'). \tag{6.3.8}$$

This is still an exact equation. We have replaced two linear differential equations by one linear differential-integral equation. Next we make some approximations.

Assuming that the modes of the field are closely spaced in frequency, we can replace the summation over $\mathbf{k}$ by an integral:

$$\sum_{\mathbf{k}} \to 2\frac{V}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2, \tag{6.3.9}$$

where V is the quantization volume. It follows from Eq. (6.1.8) that

$$|g_{\mathbf{k}}(\mathbf{r}_0)|^2 = \frac{\nu_k}{2\hbar\epsilon_0 V} \wp_{ab}^2 \cos^2\theta, \tag{6.3.10}$$

where θ is the angle between the atomic dipole moment $\wp_{ab}$ and the electric field polarization vector $\hat{\epsilon}_{\mathbf{k}}$. Equation (6.3.8) therefore becomes

$$\dot{c}_a(t) = -\frac{4\wp_{ab}^2}{(2\pi)^2 6\hbar\epsilon_0 c^3} \int_0^{\infty} d\nu_k \nu_k^3 \int_0^t dt' e^{i(\omega-\nu_k)(t-t')} c_a(t'), \tag{6.3.11}$$

where integrations over θ and ϕ have been carried out and we have used $k = \nu_k/c$. In the emission spectrum, the intensity of light associated with the emitted radiation is going to be centered about the atomic transition frequency ω. The quantity ν_k^3 varies little around $\nu_k = \omega$ for which the time integral in Eq. (6.3.11) is not negligible. We can therefore replace ν_k^3 by ω^3 and the lower limit in the ν_k integration by $-\infty$. The integral

$$\int_{-\infty}^{\infty} d\nu_k e^{i(\omega-\nu_k)(t-t')} = 2\pi\delta(t-t'), \tag{6.3.12}$$

yields the following equation for $c_a(t)$, in the Weisskopf–Wigner approximation:

$$\dot{c}_a(t) = -\frac{\Gamma}{2} c_a(t), \tag{6.3.13}$$

where the decay constant

$$\Gamma = \frac{1}{4\pi\epsilon_0} \frac{4\omega^3 \wp_{ab}^2}{3\hbar c^3}. \tag{6.3.14}$$

A solution of Eq. (6.3.13) gives

$$\rho_{aa} \equiv |c_a(t)|^2 = \exp(-\Gamma t), \tag{6.3.15}$$

i.e., an atom in the excited state $|a\rangle$ in vacuum decays exponentially in time with the lifetime $\tau = 1/\Gamma$.

During the process of spontaneous emission, the atom emits a quantum of energy equal to $E_a - E_b = \hbar\nu$. We now calculate the state of the field emitted during the spontaneous emission process.

We first calculate the coefficient $c_{b,\mathbf{k}}(t)$. Substituting the solution for $c_a(t)$ into Eq. (6.3.7) we find

$$\begin{aligned} c_{b,\mathbf{k}}(t) &= -ig_{\mathbf{k}}(\mathbf{r}_0)\int_0^t dt' e^{-i(\omega-\nu_k)t'-\Gamma t'/2} \\ &= g_{\mathbf{k}}(\mathbf{r}_0)\left[\frac{1-e^{-i(\omega-\nu_k)t-\Gamma t/2}}{(\nu_k-\omega)+i\Gamma/2}\right], \end{aligned} \tag{6.3.16}$$

so that

$$\begin{aligned} |\psi(t)\rangle &= e^{-\Gamma t/2}|a,0\rangle \\ &\quad + |b\rangle \sum_{\mathbf{k}} g_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}_0}\left[\frac{1-e^{i(\omega-\nu_k)t-\Gamma t/2}}{(\nu_k-\omega)+i\Gamma/2}\right]|1_{\mathbf{k}}\rangle. \end{aligned} \tag{6.3.17}$$

Upon introducing the field state

$$|\gamma_0\rangle = \sum_{\mathbf{k}} g_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}_0}}{(\nu_k-\omega)+i\Gamma/2}|1_{\mathbf{k}}\rangle, \tag{6.3.18}$$

for times long compared to the radiative decay $t \gg \Gamma^{-1}$ we have $|\psi\rangle \to |b\rangle|\gamma_0\rangle$. Here the index '0' in $|\gamma_0\rangle$ reminds us that this state corresponds to an atom located at position $\mathbf{r}_0$. This is a linear superposition of the single-photon states with different wave vectors associated with them.

The first-order correlation function $G^{(1)}(\mathbf{r},\mathbf{r};t,t)$ for large times is given by

$$\begin{aligned} G^{(1)}(\mathbf{r},\mathbf{r};t,t) &= \langle\psi|E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)|\psi\rangle \\ &= \langle\gamma_0|E^{(-)}(\mathbf{r},t)E^{(+)}(\mathbf{r},t)|\gamma_0\rangle \\ &= \langle\gamma_0|E^{(-)}(\mathbf{r},t)|0\rangle\langle 0|E^{(+)}(\mathbf{r},t)|\gamma_0\rangle. \end{aligned} \tag{6.3.19}$$

Here a complete set of states is inserted and since only the vacuum state survives while the other states lead to zero, we keep the vacuum state only. We have also assumed, that the field is linearly polarized, say along the x-axis. As discussed in Section 4.2, $G^{(1)}(\mathbf{r},\mathbf{r};t,t)$ is proportional to the probability of registering a photon at time t by a photodetector located at the position $\mathbf{r}$. According to Eq. (6.3.19), $G^{(1)}(\mathbf{r},\mathbf{r};t,t) = |\langle 0|E^{(+)}(\mathbf{r},t)|\gamma_0\rangle|^2$. Thus the function

$$\Psi_\gamma(\mathbf{r},t) = \langle 0|E^{(+)}(\mathbf{r},t)|\gamma_0\rangle \tag{6.3.20}$$

can be interpreted as a kind of *wave function* for a photon. This is in analogy with the corresponding wave function for particles (see Section 1.5).

From the definitions of $E^{(+)}(\mathbf{r}, t)$ and $|\gamma_0\rangle$ (Eqs. (1.1.30) and (6.3.18)), we find

$$\begin{aligned}\langle 0|E^{(+)}(\mathbf{r}, t)|\gamma_0\rangle &= \\ &= \sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k},\mathbf{k}'} \langle 0|(\nu_{k'})^{1/2} a_{\mathbf{k}'} e^{-i\nu_{k'} t + i\mathbf{k}'\cdot\mathbf{r}} g_{\mathbf{k}} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}_0}}{(\nu_k - \omega) + i\Gamma/2} |1_{\mathbf{k}}\rangle \\ &= \sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} (\nu_k)^{1/2} g_{\mathbf{k}} e^{-i\nu_k t} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} \frac{1}{(\nu_k - \omega) + i\Gamma/2}. \qquad (6.3.21)\end{aligned}$$

We now evaluate this function by first converting the sum into an integral via Eq. (6.3.9). We however do not include the factor 2 from there as the field is assumed to be polarized along the x-axis. The ϕ- and θ-integrations can be carried out by choosing a coordinate system in which the vector $\mathbf{r} - \mathbf{r}_0$ points along the z-axis, the atomic dipole moment forms an angle η with the z-axis in the x-z plane, and the wave vector $\mathbf{k}$ has components

$$\mathbf{k} = k(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}). \qquad (6.3.22)$$

The resulting expression for $\langle 0|E^{(+)}(\mathbf{r}, t)|\gamma_0\rangle$ is*

$$\begin{aligned}\langle 0|E^{(+)}(\mathbf{r}, t)|\gamma_0\rangle = &\frac{ic\wp_{ab}\sin\eta}{8\pi^2\epsilon_0\Delta r} \\ &\times \int_0^\infty dk k^2 \left(e^{ik\Delta r} - e^{-ik\Delta r}\right) \frac{e^{-i\nu_k t}}{(\nu_k - \omega) + i\Gamma/2},\end{aligned} \qquad (6.3.23)$$

where $\Delta r = |\mathbf{r} - \mathbf{r}_0|$. In the above integral the term $\exp[-i(k\Delta r + \nu_k t)]$ represents an incoming wave and we will therefore neglect it. As in the Weisskopf–Wigner theory of spontaneous emission, we assume that the quantity ν_k^2 varies little around $\nu_k = \omega$ and therefore can be replaced by ω^2 and the lower limit of integration can be extended to $-\infty$. Making these approximations we are left with the integral

$$\int_{-\infty}^{\infty} d\nu_k \frac{e^{-i\nu_k t + i\nu_k \Delta r/c}}{(\nu_k - \omega) + i\Gamma/2}.$$

This integral is evaluated by using the contour method (see Fig. 6.3). For $t < \Delta r/c$, the contour lies in the upper half plane and if $t > \Delta r/c$, in the lower half-plane. On performing the integration, we find that

* Equation (6.3.23) can be derived in a more complete and rigorous way using the method in Appendix 10.A.

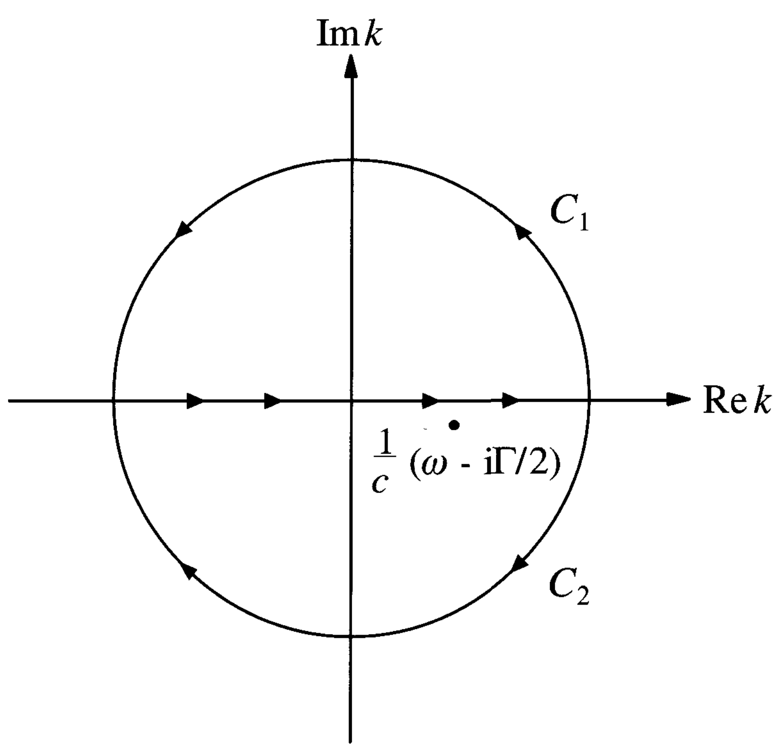


Fig. 6.3
Contours used for evaluating the integral in Eq. (6.3.23): C_1 if $t < \Delta r/c$ and C_2 if $t > \Delta r/c$.

$$\langle 0|E^{(+)}(\mathbf{r},t)|\gamma_0\rangle = \frac{\mathscr{E}_0}{\Delta r}\Theta\left(t - \frac{\Delta r}{c}\right)e^{-i(t-\frac{\Delta r}{c})(\omega - i\Gamma/2)}, \qquad (6.3.24)$$

where Θ is a unit step function and

$$\mathscr{E}_0 = -\frac{\omega^2 \wp_{ab}\sin\eta}{4\pi\epsilon_0 c^2 \Delta r}. \qquad (6.3.25)$$

We then find that

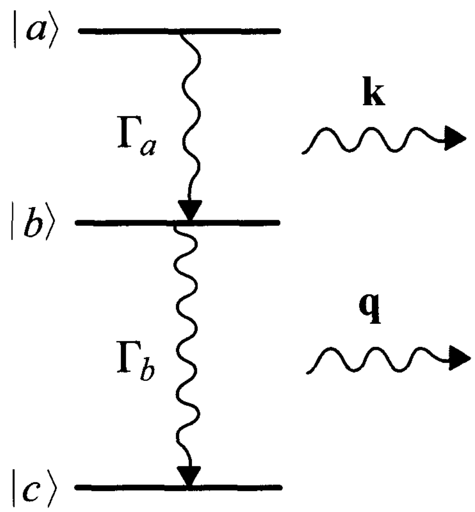
$$G^{(1)}(\mathbf{r},\mathbf{r};t,t) = \frac{|\mathscr{E}_0|^2}{|\mathbf{r}-\mathbf{r}_0|^2}\Theta\left(t - \frac{|\mathbf{r}-\mathbf{r}_0|}{c}\right)e^{-\Gamma(t-|\mathbf{r}-\mathbf{r}_0|/c)}. \qquad (6.3.26)$$

Here the step function is a manifestation of the fact that the signal cannot move faster than the speed of light.

6.4 Two-photon cascades

In this section we consider the spontaneous emission in a three-level atom in cascade configuration, as shown in Fig. 6.4. The atom in upper state $|a\rangle$ emits a $\mathbf{k}$ photon of frequency ν_k and decays to state

Fig. 6.4
Level scheme for atomic decay due to spontaneous emission in a three-level atom in cascade configuration.



$|b\rangle$ which decays to ground state $|c\rangle$, via emission of a $\mathbf{q}$ photon of frequency ν_q. The interaction picture Hamiltonian for the system is

$$\mathscr{V} = \hbar \sum_{\mathbf{k}} \left[g^*_{a,\mathbf{k}}(\mathbf{r}_0)\sigma^{(1)}_+ a_{\mathbf{k}} e^{i(\omega_{ab}-\nu_k)t} + \text{H.c.} \right] + \hbar \sum_{\mathbf{q}} \left[g^*_{b,\mathbf{q}}(\mathbf{r}_0)\sigma^{(2)}_+ a_{\mathbf{q}} e^{i(\omega_{bc}-\nu_q)t} + \text{H.c.} \right], \tag{6.4.1}$$

where $\sigma^{(1)}_+ = |a\rangle\langle b|$, $\sigma^{(2)}_+ = |b\rangle\langle c|$, and $g_{a,\mathbf{k}}(\mathbf{r}_0)$ and $g_{b,\mathbf{q}}(\mathbf{r}_0)$ are the appropriate coupling constants for $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$ transitions, respectively.

The state of the atom–field system is now described by

$$|\psi(t)\rangle = c_a(t)|a,0\rangle + \sum_{\mathbf{k}} c_{b,\mathbf{k}}|b,1_{\mathbf{k}}\rangle + \sum_{\mathbf{k},\mathbf{q}} c_{c,\mathbf{k},\mathbf{q}}|c,1_{\mathbf{k}},1_{\mathbf{q}}\rangle. \tag{6.4.2}$$

As in Section 6.3, the probability amplitudes c_a, $c_{b,\mathbf{k}}$, and $c_{c,\mathbf{k},\mathbf{q}}$ obey the equations of motion

$$\dot{c}_a = -i \sum_{\mathbf{k}} g^*_{a,\mathbf{k}}(\mathbf{r}_0) c_{b,\mathbf{k}} e^{i(\omega_{ab}-\nu_k)t}, \tag{6.4.3}$$

$$\dot{c}_{b,\mathbf{k}} = -i g_{a,\mathbf{k}}(\mathbf{r}_0) c_a e^{-i(\omega_{ab}-\nu_k)t} - i \sum_{\mathbf{q}} g^*_{b,\mathbf{q}}(\mathbf{r}_0) c_{c,\mathbf{k},\mathbf{q}} e^{i(\omega_{bc}-\nu_q)t}, \tag{6.4.4}$$

$$\dot{c}_{c,\mathbf{k},\mathbf{q}} = -i g_{b,\mathbf{q}}(\mathbf{r}_0) c_{b,\mathbf{k}} e^{-i(\omega_{bc}-\nu_q)t}. \tag{6.4.5}$$

Following the lead of Section 6.3, we recognize that, in the Weisskopf–Wigner approximation,

$$-i \sum_{\mathbf{k}} g^*_{a,\mathbf{k}}(\mathbf{r}_0) c_{b,\mathbf{k}} e^{i(\omega_{ab}-\nu_k)t} = -\frac{\Gamma_a}{2} c_a, \tag{6.4.6}$$

where Γ_a is the atomic decay rate from state $|a\rangle$ to state $|b\rangle$. Furthermore, it is clear that the second term in Eq. (6.4.4) represents decay from $|b\rangle$ to $|c\rangle$ and we may write

$$-i\sum_{\mathbf{q}} g^*_{b,\mathbf{q}}(\mathbf{r}_0)c_{c,\mathbf{k},\mathbf{q}}e^{i(\omega_{bc}-\nu_q)t} = -\frac{\Gamma_b}{2}c_{b,\mathbf{k}}, \tag{6.4.7}$$

where Γ_b is the decay rate from state $|b\rangle$ to state $|c\rangle$. Upon inserting Eqs. (6.4.6) and (6.4.7) into (6.4.3)–(6.4.5), we obtain the useful final form for the atom–field equations of motion

$$\dot{c}_a = -\frac{\Gamma_a}{2}c_a, \tag{6.4.8}$$

$$\dot{c}_{b,\mathbf{k}} = -ig_{a,\mathbf{k}}(\mathbf{r}_0)e^{-i(\omega_{ab}-\nu_k)t-\frac{\Gamma_a}{2}t} - \frac{\Gamma_b}{2}c_{b,\mathbf{k}}, \tag{6.4.9}$$

$$\dot{c}_{c,\mathbf{k},\mathbf{q}} = -ig_{b,\mathbf{q}}(\mathbf{r}_0)c_{b,\mathbf{k}}e^{-i(\omega_{bc}-\nu_q)t}, \tag{6.4.10}$$

where we have substituted $\exp(-\Gamma_a t/2)$ for $c_a(t)$ in the first term of Eq. (6.4.9).

We are most interested in the state of the field for times $t \gg \Gamma_a^{-1}$ and Γ_b^{-1}, i.e., we want to know $c_{c,\mathbf{k},\mathbf{q}}(\infty)$ as $c_a(\infty)$ and $c_{b,\mathbf{k}}(\infty)$ tend to zero.

It follows, on carrying out the simple integration implied by Eq. (6.4.9), that

$$\begin{aligned} c_{b,\mathbf{k}}(t) &= -ig_{a,\mathbf{k}}(\mathbf{r}_0)\int_0^t dt' e^{-i(\omega_{ab}-\nu_k)t'-\Gamma_a t'/2}e^{-\Gamma_b(t-t')/2} \\ &= -ig_{a,\mathbf{k}}(\mathbf{r}_0)\frac{e^{i(\nu_k-\omega_{ab})t-\Gamma_a t/2} - e^{-\Gamma_b t/2}}{i(\nu_k-\omega_{ab}) - \frac{1}{2}(\Gamma_a - \Gamma_b)}. \end{aligned} \tag{6.4.11}$$

This expression for $c_{b,\mathbf{k}}(t)$ can now be substituted into Eq. (6.4.10), and the resulting equation can be integrated to yield the following long time limit of $c_{c,\mathbf{k},\mathbf{q}}(t)$:

$$\begin{aligned} c_{c,\mathbf{k},\mathbf{q}}(\infty) &= g_{a,\mathbf{k}}g_{b,\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_0}\frac{1}{i(\nu_k-\omega_{ab}) - \frac{1}{2}(\Gamma_a-\Gamma_b)} \\ &\quad\times\left[\frac{1}{i(\nu_k+\nu_q-\omega_{ac}) - \frac{1}{2}\Gamma_a} - \frac{1}{i(\nu_q-\omega_{bc}) - \frac{1}{2}\Gamma_b}\right] \\ &= \frac{-g_{a,\mathbf{k}}g_{b,\mathbf{q}}e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_0}}{[i(\nu_k+\nu_q-\omega_{ac})-\frac{1}{2}\Gamma_a][i(\nu_q-\omega_{bc})-\frac{1}{2}\Gamma_b]}. \end{aligned} \tag{6.4.12}$$

As in the long time limit, both $c_a(t)$ and $c_{b,\mathbf{k}}(t)$ are zero, we insert $c_{c,\mathbf{k},\mathbf{q}}(\infty)$, as given by Eq. (6.4.12), into Eq. (6.4.2) and find that the state of the radiation field is given by

$$|\gamma, \phi\rangle = \sum_{\mathbf{k},\mathbf{q}} \frac{-g_{a,\mathbf{k}} g_{b,\mathbf{q}} e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_0}}{[i(\nu_k+\nu_q-\omega_{ac})-\frac{1}{2}\Gamma_a][i(\nu_q-\omega_{bc})-\frac{1}{2}\Gamma_b]} |1_\mathbf{k}, 1_\mathbf{q}\rangle, \tag{6.4.13}$$

where $|\gamma, \phi\rangle$ represents the two-photon state.

We shall make detailed use of this result in Chapter 21 when we utilize two-photon correlation functions in order to gain insight into the foundations of quantum mechanics.

6.5 Excitation probabilities for single and double photoexcitation events

In Section 4.2 we presented heuristic arguments to show that the photodetection probability is governed by the normally ordered field correlation functions. Here we derive the excitation probability for single and double photoelectron events using the atom–field interaction formalism developed in this chapter.*

Consider the interaction of linearly polarized light, described by the field operators $E^{(+)}(\mathbf{r}, t)$ and $E^{(-)}(\mathbf{r}, t)$, with an atomic system consisting of a lower level $|b\rangle$ and a set of excited levels $|a_j\rangle$ (Fig. 6.5). We assume that the atom is initially in state $|b\rangle$ and the field is in state $|i\rangle$. The interaction picture Hamiltonian, in the rotating-wave approximation, is

$$\mathscr{V} = -\sum_j \wp_{a_j b} \sigma_{a_j b} E^{(+)}(\mathbf{r}, t) \exp(i\omega_{a_j} t) + \text{H.c.} \tag{6.5.1}$$

The state of the atom–field system at time t is given by

$$\begin{aligned} |\psi(t)\rangle &= U_I(t)|\psi(0)\rangle \\ &\simeq \left[1 - \frac{i}{\hbar}\int_0^t dt' \mathscr{V}(t')\right] |b\rangle \otimes |i\rangle. \end{aligned} \tag{6.5.2}$$

The probability of exciting the atom to level $|a_j\rangle$ is found by calculating the expectation value of the projection operator $|a_j\rangle\langle a_j|$, i.e.,

$$\begin{aligned} P_j(t) &= \langle\psi(t)|a_j\rangle\langle a_j|\psi(t)\rangle \\ &= \frac{\wp_{a_j b}^2}{\hbar^2} \int_0^t \int_0^t dt_1 dt_2 \exp[i\omega_{a_j}(t_1-t_2)] \langle i|E^{(-)}(\mathbf{r}, t_2) E^{(+)}(\mathbf{r}, t_1)|i\rangle, \end{aligned} \tag{6.5.3}$$

where we substitute for $|\psi(t)\rangle$ from Eq. (6.5.2). If we want only the probability of excitation, we should sum over all excited levels $|a_j\rangle$.

* For an excellent treatment see the Les Houches lectures of Glauber [1965].

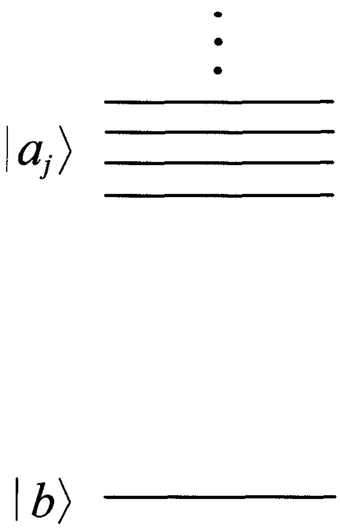


Fig. 6.5
Level scheme for photodetection. The atom makes a transition from state $|b\rangle$ to the manifold of excited states $|a_j\rangle$.

If $\wp^2_{a_jb}$ is largely independent of j, we can take $\wp^2_{a_jb} \simeq \wp^2$. Hence, for a broad-band detector the summation over j of the function $\exp[i\omega_{a_j}(t_1 - t_2)]$ introduces an effective $\delta(t_1 - t_2)$ function and we obtain

$$P(t) = \sum_j P_j(t)$$
$$= \kappa \int_0^t dt_1 \langle i|E^{(-)}(\mathbf{r}, t_1)E^{(+)}(\mathbf{r}, t_1)|i\rangle, \tag{6.5.4}$$

where κ is a constant. For mixed states, Eq. (6.5.4) becomes

$$P(t) = \kappa \int_0^t dt_1 \mathrm{Tr}\left[\rho E^{(-)}(\mathbf{r}, t_1)E^{(+)}(\mathbf{r}, t_1)\right]. \tag{6.5.5}$$

Next, we consider atoms at the points $\mathbf{r}_1$ and $\mathbf{r}_2$ and find the joint-count probability P_{12} of double photoexcitation, i.e., we want the expectation value of the photoexcitation operator

$$\left(\sum_j |a_j\rangle\langle a_j|\right)_\alpha$$

for both atoms, i.e., $\alpha = 1$ and 2. Similarly to Eq. (6.5.5), we obtain

$$P_{12} = \kappa' \int_0^t dt_1 \int_0^t dt_2$$
$$\times \mathrm{Tr}\left[\rho E^{(-)}(\mathbf{r}_1, t_1)E^{(-)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_2, t_2)E^{(+)}(\mathbf{r}_1, t_1)\right]. \tag{6.5.6}$$

Thus P_{12} is governed by the second-order correlation function of the field operators.

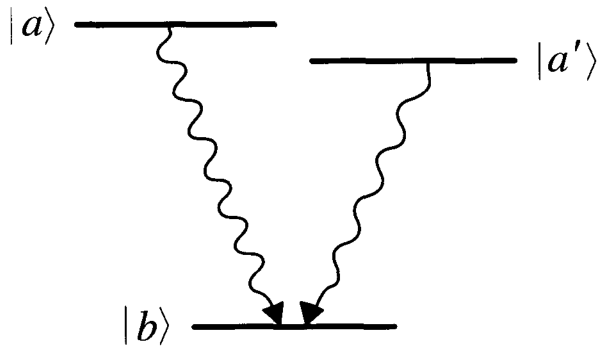


Fig. 6.6
Radiative decay of two closely lying levels $|a\rangle$ and $|a'\rangle$ to a common level $|b\rangle$.

Problems

6.1 A model sometimes considered to study the atom–field coupling in a lossless cavity is represented by the Hamiltonian

$$\mathscr{H} = \hbar\nu a^\dagger a + \hbar\omega\sigma_z + \hbar g\left[\sigma_+ a(a^\dagger a)^{1/2} + (a^\dagger a)^{1/2} a^\dagger \sigma_-\right],$$

in the usual notation. Note that the coupling is intensity dependent. Calculate the atomic inversion and discuss its evolution in terms of the various time scales, i.e., Rabi flopping time, the collapse time, and the revival time, for (a) an initial coherent state of the field and (b) an initial thermal state of the field. Note that the infinite series in the expression for inversion can be summed exactly in this case.

6.2 Calculate the population inversion for a two-level atom interacting with a single-mode quantized radiation field in the dipole and rotating-wave approximations for arbitrary time t when, at $t = 0$, the field is in a coherent state $|\alpha\rangle$, and the atomic state is $|\psi\rangle_{\text{atom}} = (|a\rangle + e^{-i\phi}|b\rangle)/\sqrt{2}$ ($|a\rangle$ and $|b\rangle$ are the upper and lower levels, respectively). Discuss the conditions under which the populations in the two levels remain 'trapped'.

6.3 Consider the atomic system shown in Fig. 6.6 with two closely spaced upper levels $|a\rangle$ and $|a'\rangle$ and a lower level $|b\rangle$. The selection rules and the energy spacing of levels $|a\rangle$ and $|a'\rangle$ is such that they interact with the same vacuum modes. The interaction of this system with a multi-mode vacuum field is described by the interaction picture Hamiltonian,

$$\mathscr{V} = \hbar \sum_{\mathbf{k}} \left[g_{\mathbf{k}}^{(ab)} a_{\mathbf{k}}^{\dagger} |b\rangle\langle a| e^{-i(\omega_{ab}-\nu_k)t} + g_{\mathbf{k}}^{(a'b)} a_{\mathbf{k}}^{\dagger} |b\rangle\langle a'| e^{-i(\omega_{a'b}-\nu_k)t} \right] + \text{H.c.}$$

Here $a_{\mathbf{k}}^{\dagger}$ is the creation operator for the mode with wave vector $\mathbf{k}$, and $\omega_{ab} = \omega_a - \omega_b$, $\omega_{a'b} = \omega_{a'} - \omega_b$. Derive the amplitude equations of motion for the three levels and show that quantum interference effects arise due to the sharing of common vacuum modes by the upper two levels.

Hint: see Zhu, Narducci, and Scully, *Phys. Rev. A* **52**, 6 (1995).

6.4 If $C = \frac{1}{2}\Delta\sigma_z + g(\sigma_+ a + a^{\dagger}\sigma_-)$ and $N = a^{\dagger}a + \sigma_+\sigma_-$, show that

$$C^2 = \frac{\Delta^2}{4} + g^2 N.$$

6.5 Prove Eq. (6.2.44).

References and bibliography

Review articles and textbooks on the two-level systems interacting with single-mode quantized field

S. Stenholm, *Phys. Rep.* **6**, 1 (1973).
M. Sargent III, M. Scully, and W. E. Lamb, Jr., *Laser Physics*, (Addison-Wesley, Reading, MA, 1974).
L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms*, (John Wiley, New York, 1975).
P. L. Knight and P. W. Milonni, *Phys. Rep.* **66**, 21 (1980).
S. Stenholm, *Foundations of Laser Spectroscopy*, (Wiley, New York, 1984).
C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms, Introduction to Quantum Electrodynamics*, (Wiley, New York 1989).
S. Haroche and D. Kleppner, *Physics Today* **42**, Jan. 24 (1989).
C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions*, (Wiley, New York 1992).
B. W. Shore and P. L. Knight, *J. Mod. Opt.* **40**, 1195 (1993).
P. Berman ed., *Cavity Quantum Electrodynamics* (Acad. Press, New York 1994).
D. F. Walls and G. J. Milburn, *Quantum Optics*, (Springer-Verlag, Berlin 1994).

General references

E. T. Jaynes and F. W. Cummings, *Proc. IEEE* **51**, 89 (1963).
R. J. Glauber, in *Quantum Optics and Electronics*, Les Houches, ed. C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York 1965).
M. O. Scully and W. E. Lamb, Jr., *Phys. Rev.* **159**, 208 (1967).
M. Tavis and F. W. Cummings, *Phys. Rev.* **188**, 692 (1969).
P. Meystre, A. Quattropani, and H. P. Baltes, *Phys. Lett.* **49A**, 85 (1974).
T. von Foerster, *J. Phys. A* **8**, 95 (1975).
J. R. Ackerhalt and K. Rzazewski, *Phys. Rev. A* **12**, 2549 (1975).
N. B. Narozhny, J. J. Sanchez-Mondragon, and J. H. Eberly, *Phys. Rev. A* **23**, 236 (1981).
P. L. Knight and P. M. Radmore, *Phys. Rev. A* **26**, 676 (1982).
K. Zaheer and M. S. Zubairy, *Phys. Rev. A* **37**, 1628 (1988).
J. Eiselt and H. Risken, *Opt. Commun.* **72**, 351 (1989); *Phys. Rev. A* **43**, 346 (1991).
T. Quang, P. L. Knight, and V. Bužek, *Phys. Rev. A* **44**, 6092 (1991).

Generalized models for quantized atom–field interactions

R. J. Cook and B. W. Shore, *Phys. Rev. A* **20**, 539 (1979).
S. Kumar and C. L. Mehta, *Phys. Rev. A* **21**, 1573 (1980).
B. Buck and C. V. Sukumar, *Phys. Lett. A* **81**, 132 (1981).
S. Singh, *Phys. Rev. A* **25**, 3206 (1982).
N. N. Bogolubov, Jr., F. Le Kien, and A. S. Shumovsky, *Phys. Lett. A* **101**, 201 (1984); *ibid.* **107**, 456 (1985).
Z. Deng, *Opt. Commun.* **54**, 222 (1985).

J. Seke, *J. Opt. Soc. Am. B* **2**, 968 (1985).
A. M. Abdel-Hafez, A. S. F. Obada, and M. M. A. Ahmad, *Phys. Rev. A* **35**, 1634 (1987).
P. Alsing and M. S. Zubairy, *J. Opt. Soc. Am. B* **4**, 177 (1987).
M. S. Iqbal, S. Mahmood, M. S. K. Razmi, and M. S. Zubairy, *J. Opt. Soc. Am. B* **5**, 1312 (1988).
R. R. Puri and R. K. Bullough, *J. Opt. Soc. Am. B* **5**, 2021 (1988).
V. Bužek and I. Jex, *J. Mod. Opt.* **36**, 1427 (1989).
M. P. Sharma, D. A. Cardimona, and A. Gavrielides, *J. Opt. Soc. Am. B* **6**, 1942 (1989).
V. Bužek, *Phys. Rev. A* **39**, 3196 (1989); *J. Mod. Opt.* **37**, 1033 (1990).
G. Adam, J. Seke, and O. Hittmair, *Phys. Rev. A* **42**, 5522 (1990).
A. Joshi and R. R. Puri, *Phys. Rev. A* **42**, 4336 (1990).
A. H. Toor and M. S. Zubairy, *Phys. Rev. A* **45**, 4951 (1992).

Collapse and revival phenomena

J. H. Eberly, N. B. Narozhny, and J. J. Sanchez-Mondragon, *Phys. Rev. Lett.* **44**, 1323 (1980).
G. Rempe and H. Walther, *Phys. Rev. Lett.* **58**, 353 (1987).
M. Fleischhauer and W. Schleich, *Phys. Rev. A* **47**, 4258 (1993).

Squeezing and photon antibunching in atom–field interactions

P. Meystre and M. S. Zubairy, *Phys. Lett. A* **89**, 390 (1982).
P. L. Knight, *Phys. Scripta* **33** (T12), 51 (1986).
N. N. Bogolubov, Jr., F. Le Kien, and A. S. Shumovsky, *Europhys. Lett.* **4**, 281 (1987).
P. K. Aravind and G. Hu, *Physica C* **150**, 427 (1988).
C. C. Gerry, *Phys. Rev. A* **37**, 2683 (1988).
J. R. Kuklinski and J. L. Madajczyk, *Phys. Rev. A* **37**, 3175 (1988).
M. S. Kim, F. A. M. De Oliveira, and P. L. Knight, *J. Mod. Optics* **37**, 659 (1990).
M. H. Mahran, *Phys. Rev. A* **42**, 4199 (1990).
M. A. Mir and M. S. K. Razmi, *Phys. Rev. A* **44**, 6071 (1991).

Coherent superposition in atom–field interactions

S. J. D. Phoenix and P. L. Knight, *Ann. Phys. (NY)* **186**, 381 (1988).
K. Zaheer and M. S. Zubairy, *Phys. Rev. A* **39**, 2000 (1989).
J. Gea-Banacloche, *Phys. Rev. Lett.* **65**, 3385 (1990).
B. Sherman and G. Kurizki, *Phys. Rev. A* **45**, 7674 (1992).

Weisskopf–Wigner approximation

V. Weisskopf and E. Wigner, *Z. Phys.* **63**, 54 (1930).
M. Sargent III, M. Scully, and W. E. Lamb, Jr., *Laser Physics*, (Addison-Wesley, Reading, MA 1974) p. 236.

W. Louisell, *Quantum Statistical Properties of Radiation*, (Wiley, New York 1974).
H. J. Kimble, A. Mezzacappa, and P. W. Milonni, *Phys. Rev. A* **31**, 3686 (1985).

Two-photon effects and cascade emission

H. Holt, *Phys. Rev. Lett.* **19**, 1275 (1967).
L. M. Narducci, M. O. Scully, G.-L. Oppo, P. Ru, and J. R. Tredicce, *Phys. Rev. A* **42**, 1630 (1990).
Y.-F. Zhu, D. J. Gauthier, and T. W. Mossberg, *Phys. Rev. Lett.* **66**, 2460 (1991).
H. Huang and J. H. Eberly, *J. Mod. Opt.* **40**, 915 (1993).