CHAPTER 9

Quantum theory of damping – Heisenberg–Langevin approach

In the previous chapter, we developed the equation of motion for a system as it evolved under the influence of an unobserved (reservoir) system. We used the density matrix approach and worked in the interaction picture. In this chapter, we consider the same problem of the system–reservoir interaction using a quantum operator approach. We again eliminate the reservoir variables. The resulting equations for the system operators include, in addition to the damping terms, the *noise* operators which produce fluctuations. These equations have the form of classical Langevin equations, which describe, for example, the Brownian motion of a particle suspended in a liquid. The Heisenberg–Langevin approach discussed in this chapter is particularly suitable for the calculation of two-time correlation functions of the system operator as is, for example, required for the determination of the natural linewidth of a laser.

We first consider the damping of the harmonic oscillator by an interaction with a reservoir consisting of many other simple harmonic oscillators. This system describes, for example, the damping of a single-mode field inside a cavity with lossy mirrors. The reservoir, in this case, consists of a large number of phonon-like modes in the mirrors. We also consider the decay of the field due to its interaction with an atomic reservoir. An interesting application of the theory of the system–reservoir interaction is the evolution of an atom inside a damped cavity. It is shown that the spontaneous transition rate of the atom can be substantially enhanced if it is placed in a resonant cavity.

9.1 Simple treatment of damping via oscillator reservoir: Markovian white noise

We consider a system consisting of a single-mode field of frequency ν and annihilation operator $a(t)$ interacting with a reservoir. The reservoir may be taken as any large collection of systems with many degrees of freedom. We assume that the reservoir consists of many oscillators (e.g., phonons, other photon modes, etc) with closely spaced frequencies ν_k and annihilation (and creation) operators $b_{\mathbf{k}}$ (and $b_{\mathbf{k}}^\dagger$). This system therefore describes the damping of a harmonic oscillator by an interaction with a reservoir consisting of many other simple harmonic oscillators. The field–reservoir system evolves in time under the influence of the total Hamiltonian

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \tag{9.1.1}$$

$$\mathscr{H}_0 = \hbar\nu a^\dagger a + \sum_{\mathbf{k}} \hbar\nu_k b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \tag{9.1.2}$$

$$\mathscr{H}_1 = \hbar \sum_{\mathbf{k}} g_{\mathbf{k}}(b_{\mathbf{k}}^\dagger a + a^\dagger b_{\mathbf{k}}). \tag{9.1.3}$$

As before, $\mathscr{H}_0$ consists of the energy of the free field and the reservoir modes, and $\mathscr{H}_1$ is the interaction energy. The field operators commute with the reservoir operators at a given time. We note that in Eq. (9.1.3) we have here made the usual rotating wave approximation.

The Heisenberg equations of motion for the operators are

$$\dot{a} = \frac{i}{\hbar}[\mathscr{H}, a] = -i\nu a(t) - i\sum_{\mathbf{k}} g_{\mathbf{k}} b_{\mathbf{k}}(t), \tag{9.1.4}$$

$$\dot{b}_{\mathbf{k}} = -i\nu_k b_{\mathbf{k}}(t) - i g_{\mathbf{k}} a(t). \tag{9.1.5}$$

We are interested in a closed equation for the harmonic oscillator operator $a(t)$. The equation for the reservoir operator $b_{\mathbf{k}}(t)$ can be formally integrated to yield

$$b_{\mathbf{k}}(t) = b_{\mathbf{k}}(0)e^{-i\nu_k t} - i g_{\mathbf{k}} \int_0^t dt' a(t') e^{-i\nu_k(t-t')}. \tag{9.1.6}$$

Here the first term represents the free evolution of the reservoir modes, whereas the second term arises from their interaction with the harmonic oscillator. The reservoir operators $b_{\mathbf{k}}(t)$ can be eliminated by substituting the formal solution of $b_{\mathbf{k}}(t)$ into Eq. (9.1.4). We find

$$\dot{a} = -i\nu a - \sum_{\mathbf{k}} g_{\mathbf{k}}^2 \int_0^t dt' a(t') e^{-i\nu_k(t-t')} + f_a(t), \tag{9.1.7}$$

$$f_a(t) = -i\sum_{\mathbf{k}} g_{\mathbf{k}} b_{\mathbf{k}}(0) e^{-i\nu_k t}. \tag{9.1.8}$$

In Eq. (9.1.7), $f_a(t)$ is a noise operator because it depends upon the reservoir operators $b_{\mathbf{k}}(0)$. The evolution of the expectation values involving the harmonic oscillator operator will therefore depend upon the fluctuations in the reservoir. The noise operator varies rapidly due to the presence of all the reservoir frequencies. The fast frequency dependence of $a(t)$ can be removed by transforming to the slowly varying annihilation operator

$$\tilde{a}(t) = a(t)e^{i\nu t}. \tag{9.1.9}$$

We see that

$$[\tilde{a}(t), \tilde{a}^\dagger(t)] = 1, \tag{9.1.10}$$

and Eq. (9.1.7) reduces to

$$\dot{\tilde{a}} = -\sum_{\mathbf{k}} g_{\mathbf{k}}^2 \int_0^t dt' \tilde{a}(t') e^{-i(\nu_k - \nu)(t-t')} + F_{\tilde{a}}(t), \tag{9.1.11}$$

$$F_{\tilde{a}}(t) = e^{i\nu t} f_a(t) = -i \sum_{\mathbf{k}} g_{\mathbf{k}} b_{\mathbf{k}}(0) e^{-i(\nu_k - \nu)t}. \tag{9.1.12}$$

The time integration in Eq. (9.1.11) is similar to that encountered in the Weisskopf–Wigner theory discussed in Section 6.3. As in the Weisskopf–Wigner approximation, the summation in Eq. (9.1.11) yields a $\delta(t - t')$ function and the integration can then be carried out. We obtain

$$\sum_{\mathbf{k}} g_{\mathbf{k}}^2 \int_0^t dt' \tilde{a}(t') e^{-i(\nu_k - \nu)(t-t')} \simeq \frac{1}{2}\mathscr{C}\tilde{a}(t), \tag{9.1.13}$$

where the damping constant

$$\mathscr{C} = 2\pi[g(\nu)]^2 D(\nu). \tag{9.1.14}$$

Here, $g(\nu) \equiv g_{\nu/c}$ is the coupling constant evaluated at $k = \nu/c$ and $D(\nu) = V\nu^2/\pi^2 c^3$ (with V being the quantization volume) is the density of states (see Eq. (1.1.26)). We can therefore replace Eq. (9.1.11) by the Langevin equation

$$\dot{\tilde{a}} = -\frac{1}{2}\mathscr{C}\tilde{a} + F_{\tilde{a}}(t), \tag{9.1.15}$$

where $F_{\tilde{a}}(t)$ is the noise operator which depends on the reservoir variables.

It is interesting to note that the presence of the noise operator in Eq. (9.1.15) is necessary to preserve the commutation relation (9.1.10) at all times. In the absence of the noise term ($F_{\tilde{a}}(t) = 0$), Eq. (9.1.15) can be solved and we get

$$\tilde{a}(t) = \tilde{a}(0)e^{-\mathscr{C}t/2}. \tag{9.1.16}$$

If the operator $\tilde{a}$ satisfies the commutation relation (9.1.10) at $t = 0$, then

$$[\tilde{a}(t), \tilde{a}^\dagger(t)] = e^{-\mathscr{C}t}, \tag{9.1.17}$$

representing a violation of the commutation relation. The noise operator with appropriate correlation properties helps to maintain the commutation relation (9.1.10) at all times. The presence of the noise term along with the damping term in Eq. (9.1.15) is a manifestation of the fluctuation–dissipation theorem of statistical mechanics, i.e., dissipation is always accompanied by fluctuations.

We suppose that the reservoir is in thermal equilibrium, so that

$$\langle b_{\mathbf{k}}(0)\rangle_R = \langle b_{\mathbf{k}}^\dagger(0)\rangle_R = 0, \tag{9.1.18}$$

$$\langle b_{\mathbf{k}}^\dagger(0) b_{\mathbf{k}'}(0)\rangle_R = \delta_{\mathbf{k}\mathbf{k}'}\bar{n}_{\mathbf{k}}, \tag{9.1.19}$$

$$\langle b_{\mathbf{k}}(0) b_{\mathbf{k}'}^\dagger(0)\rangle_R = (\bar{n}_{\mathbf{k}} + 1)\delta_{\mathbf{k}\mathbf{k}'}, \tag{9.1.20}$$

$$\langle b_{\mathbf{k}}(0) b_{\mathbf{k}'}(0)\rangle_R = \langle b_{\mathbf{k}}^\dagger(0) b_{\mathbf{k}'}^\dagger(0)\rangle_R = 0. \tag{9.1.21}$$

Using these relations with the noise operator value (9.1.12), we can evaluate various first- and second-order correlation functions involving $F_{\tilde{a}}(t)$ as follows:

(a) It follows trivially from Eq. (9.1.18) that the reservoir averages of $F_{\tilde{a}}(t)$ and its adjoint $F_{\tilde{a}}^\dagger(t)$ vanish, i.e.,

$$\langle F_{\tilde{a}}(t)\rangle_R = \langle F_{\tilde{a}}^\dagger(t)\rangle_R = 0. \tag{9.1.22}$$

(b) On using Eq. (9.1.19) we obtain

$$\begin{aligned}
\langle F_{\tilde{a}}^\dagger(t) F_{\tilde{a}}(t')\rangle_R &= \sum_{\mathbf{k}}\sum_{\mathbf{k}'} g_{\mathbf{k}} g_{\mathbf{k}'} \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}'}\rangle_R \exp[i(\nu_k - \nu)t - i(\nu_{k'} - \nu)t'] \\
&= \sum_{\mathbf{k}} g_{\mathbf{k}}^2 \bar{n}_{\mathbf{k}} \exp[i(\nu_k - \nu)(t - t')] \\
&= \int_0^\infty D(\nu_k)[g(\nu_k)]^2 \bar{n}(\nu_k) e^{i(\nu_k - \nu)(t - t')} d\nu_k.
\end{aligned} \tag{9.1.23}$$

In the last line, we have gone from a discrete representation to a continuous representation in the usual way. We can now pull out the slowly varying terms $D(\nu_k)$, $g(\nu_k)$, and $\bar{n}(\nu_k)$ at $\nu_k = \nu$ and replace the integral by a δ-function. This gives

$$\langle F_{\tilde{a}}^\dagger(t) F_{\tilde{a}}(t')\rangle_R = \mathscr{C}\bar{n}_{\text{th}}\delta(t - t'), \tag{9.1.24}$$

where $\mathscr{C}$ is given by Eq. (9.1.14) and $\bar{n}_{\text{th}} = \bar{n}(\nu_k)$. In analogy with the classical Langevin theory, we define the diffusion coefficient $D_{\tilde{a}^\dagger\tilde{a}}$ for $\tilde{a}^\dagger\tilde{a}$ through the equation

$$\langle F_{\tilde{a}}^\dagger(t)F_{\tilde{a}}(t')\rangle_R = 2\langle D_{\tilde{a}^\dagger\tilde{a}}\rangle_R\delta(t-t'). \tag{9.1.25}$$

Hence, from Eq. (9.1.24), the diffusion coefficient is given by

$$2\langle D_{\tilde{a}^\dagger\tilde{a}}\rangle_R = \mathscr{C}\bar{n}_{\text{th}}. \tag{9.1.26}$$

In a similar manner, we can show that

$$\langle F_{\tilde{a}}(t)F_{\tilde{a}}^\dagger(t')\rangle_R = \mathscr{C}(\bar{n}_{\text{th}}+1)\delta(t-t'), \tag{9.1.27}$$

$$\langle F_{\tilde{a}}(t)F_{\tilde{a}}(t')\rangle_R = \langle F_{\tilde{a}}^\dagger(t)F_{\tilde{a}}^\dagger(t')\rangle_R = 0, \tag{9.1.28}$$

so that

$$2\langle D_{\tilde{a}\tilde{a}^\dagger}\rangle_R = \mathscr{C}(\bar{n}_{\text{th}}+1), \tag{9.1.29}$$

$$\langle D_{\tilde{a}\tilde{a}}\rangle_R = \langle D_{\tilde{a}^\dagger\tilde{a}^\dagger}\rangle_R = 0. \tag{9.1.30}$$

(c)We now determine $\langle F_{\tilde{a}}^\dagger(t)\tilde{a}(t)\rangle_R$. This quantity will be needed below in the derivation of the equation of motion for $\langle\tilde{a}^\dagger\tilde{a}\rangle_R$. It follows, on solving Eq. (9.1.15), that

$$\tilde{a}(t) = \tilde{a}(0)\exp\left(-\frac{\mathscr{C}}{2}t\right) + \int_0^t dt'\exp\left[-\frac{\mathscr{C}}{2}(t-t')\right]F_{\tilde{a}}(t'). \tag{9.1.31}$$

We then obtain

$$\langle F_{\tilde{a}}^\dagger(t)\tilde{a}(t)\rangle_R = \langle F_{\tilde{a}}^\dagger(t)\rangle_R\tilde{a}(0)\exp\left(-\frac{\mathscr{C}}{2}t\right) + \int_0^t dt'\exp\left[-\frac{\mathscr{C}}{2}(t-t')\right]\langle F_{\tilde{a}}^\dagger(t)F_{\tilde{a}}(t')\rangle_R. \tag{9.1.32}$$

Here, we assumed that $F_{\tilde{a}}(t)$ and $\tilde{a}(0)$ are statistically independent. From Eqs. (9.1.22) and (9.1.24), it follows that

$$\langle F_{\tilde{a}}^\dagger(t)\tilde{a}(t)\rangle_R = \frac{\mathscr{C}}{2}\bar{n}_{\text{th}} = \langle D_{\tilde{a}^\dagger\tilde{a}}\rangle_R. \tag{9.1.33}$$

Similarly, we can show that

$$\langle\tilde{a}^\dagger(t)F_{\tilde{a}}(t)\rangle_R = \frac{\mathscr{C}}{2}\bar{n}_{\text{th}}. \tag{9.1.34}$$

These correlation functions will be employed to derive equations of motion for the field correlation functions in Section 9.3. We next consider the damping of a single-mode field via an atomic reservoir and also extend and strengthen the present oscillator reservoir treatment. The main result of these consideration is a correlation function for the noise operator which is not a delta function, thus corresponding to 'colored' noise as opposed to the white noise presented in this section.

9.2 Extended treatment of damping via atom and oscillator reservoirs: non-Markovian colored noise

In this section we extend our approach to the problem of damping, this time involving finite (i.e., not delta function) correlation times. We first assume a field damping mechanism via two-level atoms in thermal distribution, passing through the cavity. The atoms are assumed to be long lived and monoenergetic so that they interact with the field inside the cavity for a fixed duration τ. We then return to the oscillator reservoir model extending the treatment of the oscillator reservoir problem beyond the Markovian limit.

9.2.1 An atomic reservoir approach*

We here consider the damping of a single-mode field by an ensemble of atoms. The Hamiltonian for the present problem is given by

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1, \tag{9.2.1}$$

$$\mathscr{H}_0 = \hbar \nu a^\dagger a + \frac{1}{2}\hbar\nu \sum_i \sigma_z^i, \tag{9.2.2}$$

$$\mathscr{H}_1 = \hbar g \sum_i [f(t_i, t, \tau) a^\dagger \sigma_-^i + \text{H.c.}], \tag{9.2.3}$$

where σ_z^i and σ_-^i are the operators for the ith atom and $f(t_i, t, \tau)$ is a function which represents the injection of an atom at time t_i and its removal at a later time $t_i + \tau$. In this sense, $f(t_i, t, \tau)$ is a notch function which has the value

$$f(t_i, t, \tau) = \begin{cases} 1 & \text{for } t_i \le t < t_i + \tau, \\ 0 & \text{otherwise} . \end{cases} \tag{9.2.4}$$

For the sake of simplicity, we have assumed that the injected atoms are resonant with the field. Using this Hamiltonian, we write the equations for the field and atom operators in the interaction picture

$$\dot{a}(t) = -ig \sum_i f(t_i, t, \tau) \sigma_-^i(t), \tag{9.2.5}$$

$$\dot{\sigma}_-^i(t) = igf(t_i, t, \tau) \sigma_z^i a(t). \tag{9.2.6}$$

As before, we are interested in a closed equation for the operator $a(t)$. Integration of the atomic operator equation (9.2.6) yields

$$\sigma_-^i(t) = \sigma_-^i(t_i) + ig \int_{t_i}^t dt' f(t_i, t', \tau) \sigma_z^i(t') a(t'). \tag{9.2.7}$$

On substituting this expression for $\sigma_-^i(t)$ into the field operator

* The reader should consult Chapter 12 and Scully, Süssmann, and Benkert [1988] for further reading on the material of this section.

equation, we obtain

$$\dot{a} = g^2 \sum_i \int_{t_i}^{t} dt' f(t_i, t, \tau) f(t_i, t', \tau) \sigma_z^i(t') a(t') \\ -ig \sum_i f(t_i, t, \tau) \sigma_-^i(t_i). \tag{9.2.8}$$

If the field does not change appreciably during the transit time of the atoms, $a(t')$ in Eq. (9.2.8) can be replaced by $a(t)$. In a linear analysis, $\sigma_z^i(t')$ is also replaced by its value at the time of injection $\sigma_z^i(t_i)$. The resulting equation is

$$\dot{a} = -\frac{1}{2}\mathscr{C} a + F_a(t), \tag{9.2.9}$$

where

$$\mathscr{C} = -2g^2 \sum_i \int_{t_i}^{t} dt' f(t_i, t, \tau) f(t_i, t', \tau) \sigma_z^i(t_i), \tag{9.2.10}$$

$$F_a(t) = -ig \sum_i f(t_i, t, \tau) \sigma_-^i(t_i). \tag{9.2.11}$$

Here the decay constant $\mathscr{C}$ is positive as the inital inversion $\sigma_z^i(t_i)$ is negative in thermal equilibrium.

The noise operator $F_a(t)$ may be seen to have the moments

$$\langle F_a(t) \rangle = 0, \tag{9.2.12}$$

$$\langle F_a^\dagger(t) F_a(t') \rangle = g^2 \sum_{i,j} f(t_i, t, \tau) f(t_j, t', \tau) \langle \sigma_+^i(t_i) \sigma_-^j(t_j) \rangle \\ = g^2 [1 + \exp(\hbar\nu/k_B T)]^{-1} \sum_i f(t_i, t, \tau) f(t_i, t', \tau), \tag{9.2.13}$$

where we have used, with the atoms in a thermal equilibrium state at temperature T, (by solving Eqs. (8.2.10a) and (8.2.10c) in the steady state and using Eq. (8.2.5))

$$\langle \sigma_+^i(t_i) \sigma_-^j(t_j) \rangle = \delta_{ij} [1 + \exp(\hbar\nu/k_B T)]^{-1}. \tag{9.2.14}$$

After replacing the sum over i in Eq. (9.2.13) by an integral over the injection time,

$$\sum_i \rightarrow r_a \int_{-\infty}^{t} dt_i, \tag{9.2.15}$$

where r_a is the rate of injection of atoms into the cavity, we find

$$\langle F_a^\dagger(t)F_a(t')\rangle = r_a g^2\left[1+\exp\left(\frac{\hbar\nu}{k_B T}\right)\right]^{-1}\int_{-\infty}^{t} dt_i f(t_i,t,\tau)f(t_i,t',\tau). \tag{9.2.16}$$

The integration can be carried out, for example, by writing

$$f(t_i,t,\tau) = \Theta(t-t_i) - \Theta(t-\tau-t_i), \tag{9.2.17}$$

where Θ is the unit step function and using

$$\int_{-\infty}^{\infty} dt_i \Theta(t_1-t_i)\Theta(t_2-t_i) = \Theta(t_1-t_2)\int_{-\infty}^{t_2} dt_i + \Theta(t_2-t_1)\int_{-\infty}^{t_1} dt_i. \tag{9.2.18}$$

We then obtain

$$\begin{aligned}\int_{-\infty}^{t} dt_i f(t_i,t,\tau)f(t_i,t',\tau) = {}& [\Theta(t-t') - \Theta(t-t'-\tau)]\int_{-\infty}^{t'} dt_i \\ &+[\Theta(t-t') - \Theta(t-t'+\tau)]\int_{-\infty}^{t'-\tau} dt_i \\ &+[\Theta(t'-t) - \Theta(t'-t-\tau)]\int_{-\infty}^{t} dt_i \\ &+[\Theta(t'-t) - \Theta(t'-t+\tau)]\int_{-\infty}^{t-\tau} dt_i.\end{aligned} \tag{9.2.19}$$

A careful examination shows that the right hand-side of Eq. (9.2.19) is zero unless $\tau \geq |t-t'|$ in which case it is equal to $\tau - |t-t'|$. The correlation function (9.2.16) is therefore given by

$$\langle F_a^\dagger(t)F_a(t')\rangle = \begin{cases} \alpha_F(\tau-|t-t'|)/\tau^2 & \text{for } |t-t'| \leq \tau\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{9.2.20}$$

where $\alpha_F = r_a g^2\tau^2[1+\exp(\hbar\nu/k_B T)]^{-1}$. The correlation function is triangularly shaped as depicted in Fig. 9.1. This is one of the simplest examples of a 'colored' noise problem.

9.2.2 A generalized treatment of the oscillator reservoir problem*

We now present a treatment of the multi-oscillator heat bath problem. For an oscillator of momentum p and coordinate x coupled to a bath of

* This section follows the paper by Ford, Lewis, and O'Connell [1988].

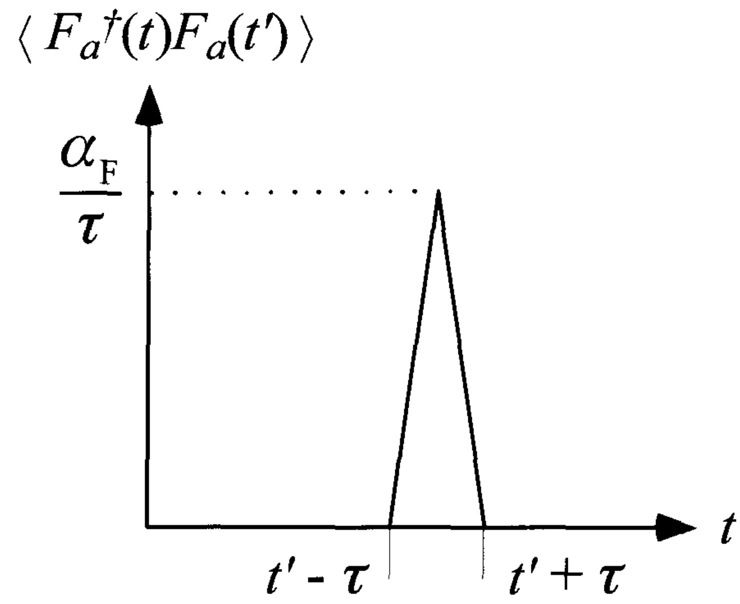

Fig. 9.1
Noise correlation function $\langle F_a^\dagger(t)F_a(t')\rangle$ as given in Eq. (9.2.20).

oscillators having momentum p_j and position q_j, the system–reservoir Hamiltonian can be written as

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}mv^2x^2 + \sum_j \frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(q_j - x)^2. \qquad (9.2.21)$$

Note that in this form the Hamiltonian (9.2.21) does not make the rotating-wave approximation. Including the normal commutation rules $[x, p] = i\hbar$ and $\left[q_j, p_k\right] = i\hbar\delta_{jk}$, we find

$$\dot{x} = \frac{1}{i\hbar}\,[x, \mathscr{H}] = \frac{p}{m}, \qquad (9.2.22a)$$

$$\dot{p} = \frac{1}{i\hbar}\,[p, \mathscr{H}] = -mv^2x + \sum_j m_j\omega_j^2(q_j - x), \qquad (9.2.22b)$$

$$\dot{q}_j = \frac{1}{i\hbar}\,\left[q_j, \mathscr{H}\right] = \frac{p_j}{m_j}, \qquad (9.2.22c)$$

$$\dot{p}_j = \frac{1}{i\hbar}\,\left[p_j, \mathscr{H}\right] = -m_j\omega_j^2(q_j - x). \qquad (9.2.22d)$$

Differentiating Eqs. (9.2.22a) and (9.2.22c) and using Eqs. (9.2.22b) and (9.2.22d), we find

$$\ddot{x}(t) = -v^2x(t) + \sum_j \frac{m_j}{m}\omega_j^2[q_j(t) - x(t)], \qquad (9.2.23a)$$

$$\ddot{q}_j(t) = -\omega_j^2[q_j(t) - x(t)]. \qquad (9.2.23b)$$

As may be verified by direct substitution, the solution for $q_j(t)$ may be written in the form

$$q_j(t) - x(t) = q_j^0(t) - \int_{-\infty}^{t} dt' \cos\left[\omega_j(t-t')\right] \dot{x}(t'), \tag{9.2.24}$$

where $q_j^0(t)$ is the solution to the problem in the absence of coupling $x = 0$

$$q_j^0(t) = q_j \cos\omega_j t + p_j \frac{\sin\omega_j t}{m_j\omega_j}, \tag{9.2.25}$$

in which q_j and p_j are the usual time-independent position and momentum operators.

Substituting (9.2.24) into (9.2.23a) we find

$$m\ddot{x}(t) + \int_{-\infty}^{t} dt' \mu(t-t')\dot{x}(t') + m\nu^2 x(t) = F(t), \tag{9.2.26}$$

where the damping function is given by

$$\mu(t-t') = \sum_j m_j\omega_j^2 \cos\left[\omega_j(t-t')\right], \tag{9.2.27a}$$

and the noise operator takes the form

$$F(t) = \sum_j m_j\omega_j^2 q_j^0(t). \tag{9.2.27b}$$

As it stands, Eq. (9.2.26) is closely related to Eq. (9.1.11). However, the problem can be extended to include memory effects by writing the following general expression for a damped oscillator

$$m\ddot{x}(t) + \int_{-\infty}^{t} dt' \mu(t-t')\dot{x}(t') + m\nu^2 x = F(t), \tag{9.2.28}$$

where

$$\begin{aligned} &\frac{1}{2}\langle F(t)F(t') + F(t')F(t)\rangle \\ &= \frac{1}{\pi}\int_0^\infty d\omega \mathrm{Re}\left[\tilde{\mu}(\omega + i0^+)\right] \hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) \cos\left[\omega(t-t')\right], \end{aligned} \tag{9.2.29}$$

with $\tilde{\mu}$ being the Fourier transform of $\mu(t)$.

Now for the case of constant damping, which is the one of most interest to us, $\mathrm{Re}\left[\tilde{\mu}(\omega + i0^+)\right] = \Gamma$ and the correlation function takes the form

$$\begin{aligned} &\frac{1}{2}\langle F(t)F(t') + F(t')F(t)\rangle \\ &= \frac{\Gamma}{\pi}\int_0^\infty d\omega \hbar\omega \coth\left(\frac{\hbar\omega}{2k_B T}\right) \cos\left[\omega(t-t')\right] \\ &= \Gamma k_B T \frac{d}{dt} \coth\left[\frac{\pi k_B T(t-t')}{\hbar}\right]. \end{aligned} \tag{9.2.30}$$

We note that Eq. (9.2.30), while going to $\delta(t-t')$ in the limit, in general goes beyond the Markovian approximation, i.e., it implies colored noise.

9.3 Equations of motion for the field correlation functions

We can now derive the mean motion of $\tilde{a}(t)$ and of the number operator $\tilde{a}^\dagger\tilde{a}$. Since $\langle F_{\tilde{a}}(t)\rangle_R = 0$, it follows from Eq. (9.1.15), that

$$\frac{d}{dt}\langle\tilde{a}(t)\rangle_R = -\frac{1}{2}\mathscr{C}\langle\tilde{a}(t)\rangle_R. \tag{9.3.1}$$

Here, we see that the mean value of the system operator goes to zero in time. Note that Eq. (9.3.1) is only averaged over the reservoir coordinates It remains an operator in the field coordinates.

The mean time development of the field number operator is

$$\begin{aligned}\frac{d}{dt}\langle\tilde{a}^\dagger(t)\tilde{a}(t)\rangle_R &= \left\langle\frac{d\tilde{a}^\dagger(t)}{dt}\tilde{a}(t)\right\rangle_R + \left\langle\tilde{a}^\dagger(t)\frac{d\tilde{a}(t)}{dt}\right\rangle_R \\ &= -\mathscr{C}\langle\tilde{a}^\dagger(t)\tilde{a}(t)\rangle_R + \langle F^\dagger_{\tilde{a}}(t)\tilde{a}(t)\rangle_R + \langle\tilde{a}^\dagger(t)F_{\tilde{a}}(t)\rangle_R \\ &= -\mathscr{C}\langle\tilde{a}^\dagger(t)\tilde{a}(t)\rangle_R + \mathscr{C}\bar{n}_{\text{th}}.\end{aligned} \tag{9.3.2}$$

Thus, the steady-state value of the number operator $\langle\tilde{a}^\dagger(t)\tilde{a}(t)\rangle_R$ is $\bar{n}_{\text{th}}$ (times the field identity operator); this is nonzero in contrast to $\langle\tilde{a}^\dagger(t)\rangle_R$ and $\langle\tilde{a}(t)\rangle_R$, which decay to zero in time according to Eq. (9.3.1).

In a similar manner, it can be shown that

$$\frac{d}{dt}\langle\tilde{a}(t)\tilde{a}^\dagger(t)\rangle_R = -\mathscr{C}\langle\tilde{a}(t)\tilde{a}^\dagger(t)\rangle_R + \mathscr{C}(\bar{n}_{\text{th}} + 1). \tag{9.3.3}$$

On combining Eqs. (9.3.2) and (9.3.3), we see that the commutator $[\tilde{a}(t), \tilde{a}^\dagger(t)]$ retains its unity reservoir average in time instead of decaying to zero.

Using the same arguments as given for the derivation for the equations of motion for $\langle\tilde{a}(t)\rangle_R$ and $\langle\tilde{a}^\dagger(t)\tilde{a}(t)\rangle_R$, Eqs. (9.3.1) and (9.3.2), we can show that for arbitrary products of the creation and annihilation operators,

$$\frac{d}{dt}\langle(\tilde{a}^\dagger)^m\tilde{a}^n\rangle_R = -\frac{\mathscr{C}}{2}(m+n)\langle(\tilde{a}^\dagger)^m\tilde{a}^n\rangle_R + \mathscr{C}mn\bar{n}_{\text{th}}\langle(\tilde{a}^\dagger)^{m-1}\tilde{a}^{n-1}\rangle_R. \tag{9.3.4}$$

In terms of the operators a and $a^\dagger$ (Eq. (9.1.9)) this equation reads

$$\begin{aligned}\frac{d}{dt}\langle(a^\dagger)^m(a)^n\rangle_R = &\left[i\nu(m-n) - \frac{\mathscr{C}}{2}(m+n)\right]\langle(a^\dagger)^m a^n\rangle_R \\ &+ \mathscr{C}mn\bar{n}_{\text{th}}\langle(a^\dagger)^{m-1}a^{n-1}\rangle_R.\end{aligned} \tag{9.3.5}$$

This equation, in a general way, describes the effect of the reservoir.

As mentioned earlier, the present Heisenberg–Langevin approach to the quantum theory of damping is particularly suited for the calculation of multi-time correlation functions. This can be appreciated by considering the simple example of the damping of the field of frequency ν inside the cavity at the rate $\mathscr{C} = \nu/Q$. Here Q is the quality factor of the cavity.

The field operator $\tilde{a}(t) = a(t)\exp(i\nu t)$ obeys the equation

$$\dot{\tilde{a}} = -\frac{\nu}{2Q}\tilde{a} + F_{\tilde{a}}(t), \tag{9.3.6}$$

which can be solved to yield (with $\tau > 0$)

$$\tilde{a}(t_i+\tau) = \tilde{a}(t_i)\exp\left(-\frac{\nu}{2Q}\tau\right) + \int_{t_i}^{t_i+\tau} dt' \exp\left[-\frac{\nu}{2Q}(t_i+\tau-t')\right] F_{\tilde{a}}(t'). \tag{9.3.7}$$

It follows, on using $\langle\tilde{a}^\dagger(t_i)F_{\tilde{a}}(t')\rangle_R = \langle\tilde{a}^\dagger(t_i)\rangle_R\langle F_{\tilde{a}}(t')\rangle_R = 0$, that

$$\langle\tilde{a}^\dagger(t_i)\tilde{a}(t_i+\tau)\rangle_R = \langle\tilde{a}^\dagger(t_i)\tilde{a}(t_i)\rangle_R \exp\left(-\frac{\nu}{2Q}\tau\right), \tag{9.3.8}$$

i.e., the field correlation function decays exponentially with time. The field spectrum can be obtained by taking the Fourier transform of the correlation function

$$\langle a^\dagger(t_i)a(t_i+\tau)\rangle_R = \langle\tilde{a}^\dagger(t_i)\tilde{a}(t_i+\tau)\rangle_R e^{-i\nu\tau}$$

$$= \langle n\rangle \exp\left(-i\nu\tau - \frac{\nu}{2Q}\tau\right), \tag{9.3.9}$$

(9.3.10)

where $\langle n\rangle$ is the mean number of photons at the initial time t_i. We then obtain (see Eq. (4.3.14))

$$S(\omega) = \frac{1}{\pi}\mathrm{Re}\int_0^\infty \langle a^\dagger(t)a(t+\tau)\rangle_R e^{i\omega\tau}d\tau$$

$$= \frac{\langle n\rangle}{\pi}\frac{\nu/2Q}{(\omega-\nu)^2+(\nu/2Q)^2}. \tag{9.3.11}$$

This is a Lorentzian distribution centered at $\omega = \nu$ with half-width $\nu/2Q$.

An approximate expression of the mode density of the empty cavity, $D_c(\omega)$, is obtained by dividing $S(\omega)$ by $\langle n\rangle$, i.e.,

$$D_c(\omega) = \frac{1}{\pi}\frac{\nu/2Q}{(\omega-\nu)^2+(\nu/2Q)^2}. \tag{9.3.12}$$

The density of states inside the cavity is therefore significantly different from its value in free space (see Eq. (1.1.26)).

9.4 Fluctuation–dissipation theorem and the Einstein relation

We now make a connection between the present quantum Langevin approach and the classical approach. In Section 9.1 we derived the second-order correlation function of the Langevin noise $F_{\tilde{a}}(t)$

$$\langle F_{\tilde{a}}^{\dagger}(t)F_{\tilde{a}}(t')\rangle_R = \mathscr{C}\bar{n}_{\text{th}}\delta(t-t'). \tag{9.4.1}$$

On integrating both sides, we obtain

$$\mathscr{C} = \frac{1}{\bar{n}_{\text{th}}}\int_{-\infty}^{\infty}\langle F_{\tilde{a}}^{\dagger}(t)F_{\tilde{a}}(t')\rangle_R dt'. \tag{9.4.2}$$

This states that the system damping $\mathscr{C}$ is determined from the fluctuating forces of the reservoir. Thus the fluctuations induced by the reservoir give rise to dissipation in the system. This is one formulation of the fluctuation–dissipation theorem.

Next we make use of Eqs. (9.1.15) and (9.1.26) to rewrite Eq. (9.3.2) as follows

$$\begin{aligned}2\langle D_{\tilde{a}^{\dagger}\tilde{a}}\rangle_R = {}& \frac{d}{dt}\langle\tilde{a}^{\dagger}(t)\tilde{a}(t)\rangle_R - \left\langle\left[\frac{d\tilde{a}^{\dagger}}{dt} - F_{\tilde{a}}^{\dagger}(t)\right]\tilde{a}(t)\right\rangle_R \\ & - \left\langle\tilde{a}^{\dagger}(t)\left[\frac{d\tilde{a}}{dt} - F_{\tilde{a}}(t)\right]\right\rangle_R.\end{aligned} \tag{9.4.3}$$

This is the Einstein relation to determine the diffusion constant. We have derived this relation for the damped harmonic oscillator problem. It can, however, be shown that this relation is valid for many general system–reservoir problems. It can be similarly shown that

$$\begin{aligned}2\langle D_{\tilde{a}\tilde{a}^{\dagger}}\rangle_R = {}& \frac{d}{dt}\langle\tilde{a}(t)\tilde{a}^{\dagger}(t)\rangle_R - \left\langle\tilde{a}(t)\left[\frac{d\tilde{a}^{\dagger}}{dt} - F_{\tilde{a}}^{\dagger}(t)\right]\right\rangle_R \\ & - \left\langle\left[\frac{d\tilde{a}}{dt} - F_{\tilde{a}}(t)\right]\tilde{a}^{\dagger}(t)\right\rangle_R.\end{aligned} \tag{9.4.4}$$

The Einstein relation relates the *drift* terms $[d\tilde{a}/dt - F_{\tilde{a}}(t)]$ and $[d\tilde{a}^{\dagger}/dt - F_{\tilde{a}}^{\dagger}(t)]$ to the diffusion coefficients. In many problems of interest, this relation provides an extremely simple way to calculate the diffusion constant.

The Einstein relation can be employed to determine the diffusion coefficients from the density matrix equations in a straightforward manner. In order to indicate the procedure, we consider the simple example of Eq. (8.3.2) which governs the damping of the field by an interaction with a thermal reservoir. It follows from this equation that

$$\left\langle \frac{da}{dt} \right\rangle = \text{Tr}(a\dot{\rho}) = -\frac{\mathscr{C}}{2}\langle a\rangle, \tag{9.4.5}$$

$$\left\langle \frac{da^\dagger}{dt} \right\rangle = -\frac{\mathscr{C}}{2}\langle a^\dagger\rangle, \tag{9.4.6}$$

$$\frac{d}{dt}\langle a^\dagger a\rangle = -\mathscr{C}(\langle a^\dagger a\rangle - \bar{n}_{\text{th}}), \tag{9.4.7}$$

where, in deriving these equations, we used the cyclic property of the trace (i.e., $\text{Tr}(ABC) = \text{Tr}(CAB)$, etc) and the commutation relation $[a, a^\dagger] = 1$. Now the quantities $[da/dt - F_a(t)]$ and $[da^\dagger/dt - F_a^\dagger(t)]$ can be obtained from Eqs. (9.4.5) and (9.4.6), respectively, by removing the expectation value sign on the right-hand side. We then obtain

$$\left[\frac{da}{dt} - F_a(t)\right] = -\frac{\mathscr{C}}{2}a, \tag{9.4.8}$$

$$\left[\frac{da^\dagger}{dt} - F_a^\dagger(t)\right] = -\frac{\mathscr{C}}{2}a^\dagger. \tag{9.4.9}$$

On substituting Eqs. (9.4.7)–(9.4.9) into Eq. (9.4.3), we get

$$2\langle D_{a^\dagger a}\rangle = \mathscr{C}\bar{n}_{\text{th}}, \tag{9.4.10}$$

in agreement with Eq. (9.1.26).

9.5 Atom in a damped cavity

A very simple application of the mathematical framework developed in this chapter is the study of the evolution of a single two-level atom initially prepared in the upper level $|a\rangle$ of the transition resonant with the cavity mode. In particular, it is seen that the spontaneous emission rate of the atom inside a resonant cavity is substantially enhanced over its free-space value. The enhancement factor can be derived rigorously from a quantum mechanical analysis where the cavity damping is considered via interaction of the single-mode field with a reservoir consisting of a large number of simple harmonic oscillators. First, we present an heuristic argument to understand this interesting phenomenon.

We recall that, in Section 6.3, we considered the spontaneous emission of an atom in free space, so that the atom interacts with a continuum of modes of the electromagnetic field. The decay rate Γ, as given by Eq. (6.3.14) can be rewritten as

$$\Gamma = 2\pi \langle |g(\omega)|^2 \rangle D(\omega), \tag{9.5.1}$$

where angle brackets represent an angular average, $g(\omega)$ is the vacuum Rabi frequency, and $D(\omega) = V\omega^2/\pi^2 c^3$ is the density of states at the atomic transition frequency ω. The spontaneous decay rate is therefore proportional to the density of states. The mode structure of the vacuum field is dramatically altered in a cavity whose size is comparable to the wavelength. In a cavity of quality factor Q, the mode density $D_c(\omega)$ can be approximated by the Lorentzian (Eq. (9.3.12))

$$D_c(\omega) = \frac{1}{\pi} \frac{\nu/2Q}{(\omega - \nu)^2 + (\nu/2Q)^2}. \tag{9.5.2}$$

The spontaneous decay rate of the atom inside the cavity is therefore obtained by replacing $D(\omega)$ by $D_c(\omega)$ in Eq. (9.5.1)

$$\Gamma_c = 2\pi \langle |g(\omega)|^2 \rangle D_c(\omega). \tag{9.5.3}$$

For a cavity tuned near the atomic resonance frequency, we have $D_c(\omega) \simeq 2Q/\pi\omega$ and

$$\Gamma_c = \frac{2\pi}{3} \left(\frac{\omega \wp_{ab}^2}{2\hbar\epsilon_0 V} \right) \left(\frac{2Q}{\pi\omega} \right) = \Gamma Q \left(\frac{2\pi c^3}{V\omega^3} \right). \tag{9.5.4}$$

Thus, apart from the geometrical factor of order unity (for the lowest cavity mode $\omega = \pi c/L$, where L is the length of the side of the cavity, this factor is equal to $2/\pi^2$), the spontaneous decay rate inside the cavity is enhanced by a factor Q over its free-space value.

Another simple interpretation of the spontaneous emission enhancement can be given in terms of the image charges. We can simulate the effect of the cavity mirrors on the evolution of the atom by replacing them by the Q images of the atoms in these mirrors. As the cavity is resonant with the atomic transition, all the dipoles of these images are in phase with the atomic dipole. They therefore act as Q aligned antenna in phase. A given antenna in this array radiates Q times faster than an isolated antenna. The atomic energy is therefore dissipated Q times faster than in free space.

We now turn to a rigorous derivation of the atomic decay in a damped cavity. We consider a system of a two-level atom interacting with a single-mode electromagnetic field inside a cavity. The cavity is coupled to a thermal reservoir through the walls of the cavity. The atom–field reservoir Hamiltonian is therefore

$$\mathscr{H} = \mathscr{H}_F + \mathscr{H}_A + \mathscr{H}_{AF} + \mathscr{H}_R + \mathscr{H}_{FR}, \tag{9.5.5}$$

$$\mathscr{H}_F = \hbar\nu a^\dagger a, \tag{9.5.6}$$

$$\mathscr{H}_A = \frac{1}{2}\hbar\nu\sigma_z, \tag{9.5.7}$$

$$\mathscr{H}_{AF} = \hbar g(\sigma_+ a + a^\dagger\sigma_-), \tag{9.5.8}$$

$$\mathscr{H}_R = \sum_{\mathbf{k}} \hbar\nu_k b_{\mathbf{k}}^\dagger b_{\mathbf{k}}, \tag{9.5.9}$$

$$\mathscr{H}_{FR} = \hbar\sum_{\mathbf{k}} g_{\mathbf{k}}(b_{\mathbf{k}}^\dagger a + a^\dagger b_{\mathbf{k}}). \tag{9.5.10}$$

Here $\mathscr{H}_F$ and $\mathscr{H}_A$ are the free field and atom Hamiltonians, respectively, $\mathscr{H}_{AF}$ represents the interaction of the single-mode cavity field with the atom, $\mathscr{H}_R$ is the energy of the reservoir modes and $\mathscr{H}_{FR}$ represents the interaction of the field with the reservoir. For transmission losses, the reservoir modes correspond to the vacuum modes that enter the cavity through partially transmitting mirrors. We shall assume the reservoir modes to be in thermal equilibrium at temperature T.

The quantities of interest in the system are the energy of the field $\langle a^\dagger a\rangle$ and the atomic inversion $\langle\sigma_z\rangle$. The equation of motion for any operator of the form $(a^\dagger)^m a^n O_A$, (where O_A is an atomic operator, e.g., $\sigma_+, \sigma_-, \sigma_z$) is given by

$$\frac{d}{dt}[(a^\dagger)^m a^n O_A] = -\frac{i}{\hbar}[(a^\dagger)^m a^n O_A, \mathscr{H}_F + \mathscr{H}_A + \mathscr{H}_{AF}] + \left\langle \frac{d}{dt}[(a^\dagger)^m a^n] \right\rangle_R O_A, \tag{9.5.11}$$

where $\langle d[(a^\dagger)^m a^n]/dt\rangle_R$ is given by Eq. (9.3.5). Using this equation, we can derive the following equations of motion for $\langle a^\dagger a\rangle$ and $\langle\sigma_z\rangle$:

$$\frac{d\langle a^\dagger a\rangle}{dt} = ig\langle\sigma_+ a - a^\dagger\sigma_-\rangle - \mathscr{C}\langle a^\dagger a\rangle + \mathscr{C}\bar{n}_{\text{th}}, \tag{9.5.12}$$

$$\frac{d\langle\sigma_z\rangle}{dt} = -2ig\langle\sigma_+ a - a^\dagger\sigma_-\rangle. \tag{9.5.13}$$

The angle brackets denote the reservoir as well as the quantum mechanical average. These equations involve the average of the Hermitian operator $\langle\sigma_+ a - \sigma_- a^\dagger\rangle$ whose equation of motion in turn involves the quantity $\langle a^\dagger\sigma_z a\rangle$ and so on. In general, we get an infinite set of equations which may not be analytically solvable. However, the situation is considerably simpler if initially the atom is in the excited state $|a\rangle$, the field inside the cavity is in the vacuum state $|0\rangle$, and the cavity is at zero temperature ($\bar{n}_{\text{th}} = 0$). There can be at most one photon in the field and the state of the field inside the cavity at any time t will be a linear superposition of the vacuum state $|0\rangle$ and the one-photon state $|1\rangle$. The expectation value of the operators involving quadratic or higher powers in the field operators a and $a^\dagger$, e.g., $\langle (a^\dagger)^2 \sigma_z a^2 \rangle$, are therefore zero at all times. Under these conditions, we obtain the following closed set of equations

$$\frac{d\langle a^\dagger a \rangle}{dt} = gA_1 - \mathscr{C}\langle a^\dagger a \rangle, \tag{9.5.14}$$

$$\frac{d\langle \sigma_z \rangle}{dt} = -2gA_1, \tag{9.5.15}$$

$$\frac{dA_1}{dt} = g\langle \sigma_z \rangle + 2gA_2 + g - \frac{\mathscr{C}}{2}A_1, \tag{9.5.16}$$

$$\frac{dA_2}{dt} = -gA_1 - \mathscr{C}A_2, \tag{9.5.17}$$

where

$$A_1 = i\langle \sigma_+ a - a^\dagger \sigma_- \rangle, \tag{9.5.18}$$

$$A_2 = \langle a^\dagger \sigma_z a \rangle. \tag{9.5.19}$$

It may be noted that, in Eq. (9.5.17), we neglected the term proportional to $\langle \sigma_+ a^\dagger a^2 - (a^\dagger)^2 a \sigma_- \rangle$ in light of the above argument. The four equations (9.5.14)–(9.5.17) can be solved using, for example, the Laplace transform method. The resulting solutions for $\langle a^\dagger a \rangle_t$ and $\langle \sigma_z \rangle_t$, subject to the initial conditions $\langle a^\dagger a \rangle_0 = A_1(0) = A_2(0) = 0$ and $\langle \sigma_z \rangle_0 = 1$ are

$$\langle a^\dagger a \rangle_t = -\frac{8g^2 e^{-\mathscr{C}t/2}}{\mathscr{C}^2 - 16g^2}\left\{1 - \cosh\left[(\mathscr{C}^2 - 16g^2)^{1/2} t/2\right]\right\}, \tag{9.5.20}$$

$$\begin{aligned}\langle \sigma_z \rangle_t = -1 + \frac{4e^{-\mathscr{C}t/2}}{(\mathscr{C}^2 - 16g^2)}\Bigg\{ & -4g^2 \\ & + \left[\frac{\mathscr{C}^2}{4} - 2g^2 + \frac{\mathscr{C}}{4}(\mathscr{C}^2 - 16g^2)^{1/2}\right] \times e^{(\mathscr{C}^2 - 16g^2)^{1/2} t/2} \\ & + \left[\frac{\mathscr{C}^2}{4} - 2g^2 - \frac{\mathscr{C}}{4}(\mathscr{C}^2 - 16g^2)^{1/2}\right] \times e^{-(\mathscr{C}^2 - 16g^2)^{1/2} t/2}\Bigg\}.\end{aligned} \tag{9.5.21}$$

In Fig. 9.2, the probability of the atom being in the upper level $P_a = (1 + \langle \sigma_z \rangle)/2$ is plotted for different values of $\mathscr{C}/4g$. Here we see a transition from damped Rabi oscillations to an overdamped situation.

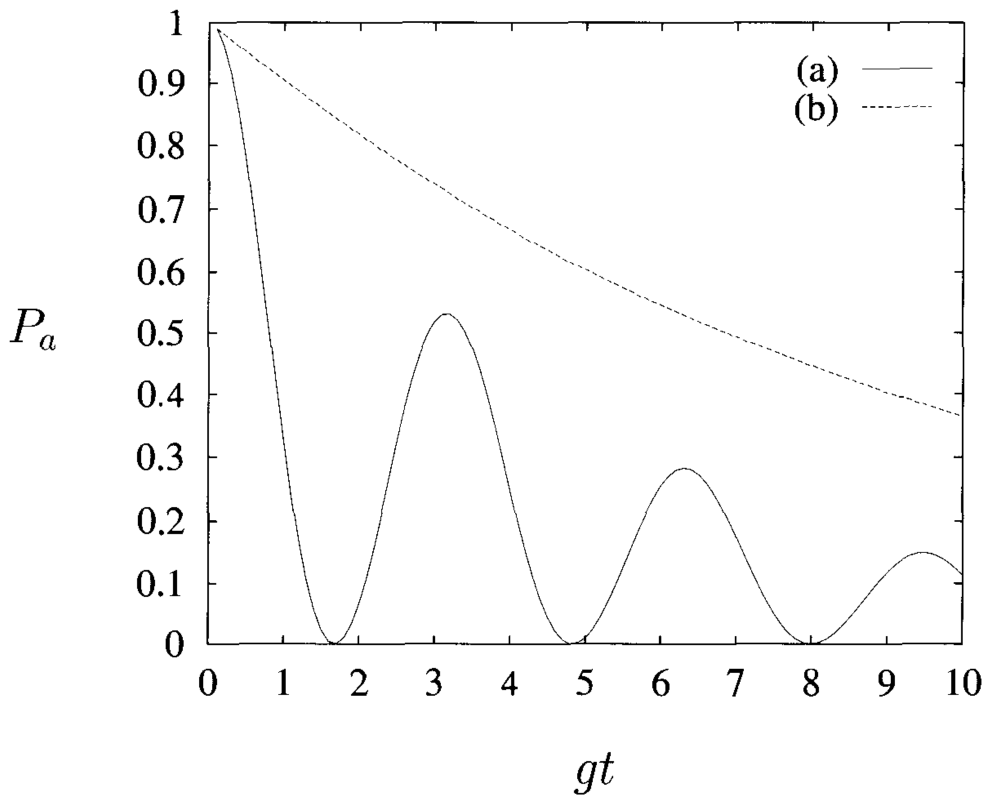

Fig. 9.2
A plot of P_a versus dimensionless time gt for (a) $\mathscr{C}/4g = 0.1$ and (b) $\mathscr{C}/4g = 10$.

This different behavior can be seen easily by considering two limiting cases of Eq. (9.5.21). When $\mathscr{C} \ll 4g$, the atomic inversion $\langle\sigma_z(t)\rangle$ and the probability P_a take the simple forms

$$\langle\sigma_z(t)\rangle = -1 + e^{-\mathscr{C}t/2}[1 + \cos(2gt)], \tag{9.5.22}$$

$$P_a(t) = \frac{e^{-\mathscr{C}t/2}}{2}[1 + \cos(2gt)]. \tag{9.5.23}$$

These damped Rabi oscillations are at the frequency $2g$. In the opposite limit $\mathscr{C} \gg 4g$, we obtain

$$\langle\sigma_z(t)\rangle = -1 + 2e^{-(4g^2t/\mathscr{C})}, \tag{9.5.24}$$

and

$$P_a(t) = e^{-(4g^2t/\mathscr{C})}, \tag{9.5.25}$$

i.e., the atom decays exponentially with a damping constant

$$\Gamma_c = \frac{4g^2}{\mathscr{C}} = \left(\frac{1}{4\pi\epsilon_0}\frac{4\nu^3\wp_{ab}^2}{3\hbar c^3}\right)\left(\frac{\nu}{\mathscr{C}}\right)\left(\frac{6\pi c^3}{V\nu^3}\right). \tag{9.5.26}$$

Apart from a trivial factor of 3, this expression is identical to Eq. (9.5.4), which was obtained using a heuristic argument based on the density of states. The factor of 3 disappears if, in Eq. (9.5.26), we replace g^2 by its average value over different orientations.

Problems

9.1 A single mode of frequency ν interacts with a thermal reservoir. The evolution of the field–reservoir system is described by the Langevin equation

$$\dot{\tilde{a}} = -\frac{1}{2}\mathscr{C}\tilde{a} + F_{\tilde{a}}(t),$$

where $\tilde{a}(t) = a(t)e^{i\nu t}$; a is the destruction operator for the field mode. Calculate the variance $(\Delta X_1)^2$ (with $X_1 = (\tilde{a} + \tilde{a}^\dagger)/2$) at a time t in terms of the variance at the initial time $t = 0$.

9.2 Find the correlation function $\langle F_a^\dagger(t)F_a(t')\rangle$ in Eq. (9.2.13) for

$$f(t_i, t, \tau) = \begin{cases} e^{-\Gamma(t-t_i)} & \text{for } t_i \leq t < t_i + \tau\,, \\ 0 & \text{otherwise.} \end{cases}$$

9.3 Calculate the second-order correlation functions

$$\langle F_{\tilde{a}}^\dagger(t)F_{\tilde{a}}(t')\rangle_R\,, \qquad \langle F_{\tilde{a}}(t)F_{\tilde{a}}^\dagger(t')\rangle_R\,,$$
$$\langle F_{\tilde{a}}(t)F_{\tilde{a}}(t')\rangle_R, \text{ and } \langle F_{\tilde{a}}^\dagger(t)F_{\tilde{a}}^\dagger(t')\rangle_R$$

of the Langevin operator for a multi-mode squeezed vacuum reservoir.

9.4 Derive the equation of motion for arbitrary products of creation and destruction operators $\langle (a^\dagger)^m a^n\rangle$ for (a) a thermal reservoir and (b) a squeezed reservoir.

9.5 Consider the reservoir in a squeezed vacuum state. Use the equation of motion for the density matrix for the field mode and the Einstein relation to calculate the diffusion coefficient $D_{\tilde{a}\tilde{a}^\dagger}$. Verify your results from Langevin theory.

References and bibliography

Operator treatment of quantum theory of damping and related problems

M. Lax, *Phys. Rev.* **145**, 110 (1966).
G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **37**, 4419 (1988).
M. O. Scully, G. Süssmann, and C. Benkert, *Phys. Rev. Lett.* **60**, 1014 (1988).

Atom in a damped cavity

E. M. Purcell, *Phys. Rev.* **69**, 681 (1946).
R. J. Cook and P. W. Milonni, *Phys. Rev.* **A35**, 5081 (1987).
S. Haroche, in *High Resolution Laser Spectroscopy*, ed. K. Shimoda (Springer-Verlag, Berlin 1976).
D. Kleppner, *Phys. Rev. Lett.* **47**, 233 (1981).
S. Sachdev, *Phys. Rev. A* **29**, 2627 (1984).
S. Haroche and J. M. Raimond, in *Advances in Atomic and Molecular Physics*, Vol. 20, ed. D. Bates and B. Bederson (Academic Press, New York 1985), p. 347.
J. A. Gallas, G. Leuchs, H. Walther, and H. Figger, in *Advances in Atomic and Molecular Physics*, Vol. 20, ed. D. Bates and B. Bederson (Academic Press, New York 1985), p. 413.

Experiments on enhanced or inhibited spontaneous emission in a cavity

P. Goy, J. M. Raimond, M. Gross, and S. Haroche, *Phys. Rev. Lett.* **50**, 1903 (1983).
G. Gabrielse, R. Van Dyck, Jr., P. Schwinberg, and H. Dehmelt, *Bull. Am. Phys. Soc.* **29**, 926 (1984).