Statistical Physics IV

École Polytechnique Fédérale de Lausanne (EPFL)

Final Exam

Exam duration: 180 minutes

- No notes are allowed during the exam.
- No calculators are allowed.
- Some useful physical constants:
  - Boltzmann constant: $k_B = 1.38 \times 10^{-23}$ [SI]
  - Planck constant: $h = 6.625 \times 10^{-34}$ [SI]
  - Speed of light: $c = 3 \times 10^{8}$ [m/s]

1. Part A (3pt/problem) - short questions

Please answer the following questions in brief and explain the concepts. (*Estimated time: 90 min*)

1. Consider an RC circuit (Fig. **??**). The capacitor is lossless and the resistor is at temperature T. Write down the Langevin equation, describing the stochastic dynamics of the voltage of the capacitor, V. Without writing the equations, explain how would the fluctuation-dissipation theorem manifest in this circuit? In particular, consider the following scenario: The capacitor is charged to a finite voltage value and at $t = 0$ it is connected to the resistor. Sketch the energy stored in the capacitor as a function of time.

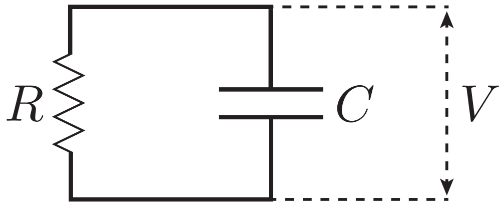

Figure 1: An RC circuit (question 1)

2. For a system connected to a reservoir at temperature T, consider two states, A and B, with the free energy difference of $\Delta F = F_A - F_B$. What does the Jarzynski equality tell us about this system and the work being done on it on an ensemble of trajectories from A to B? For an experimentalist, how is this relation useful for measuring ΔF?

3. Consider a 2D continuous time random walk characterized by the update formula $P(y, \Delta t) = f(y)\Phi(\Delta t)$, and the scaling of its mean squared displacement (MSD) $\langle r^2(t) \rangle$ to time, where $r(t)$ is the displacement from the origin at time t. Write down one example of $f(y)$ for ordinary diffusion, where the MSD scales linearly in time. Write down another example of $f(y)$ for anomalous diffusion, where the MSD obeys superlinear scaling in time.

4. Write down the quantum master equation in the Lindbard form. For a quantum harmonic oscillator coupled to a finite temperature bath, with average phonon number $\bar{n}$, write down the Lindbard operators for energy relaxation and dephasing, and explain the physical meaning of each term.

5. In the context of reservoir engineering, which jump operators are required to relax a harmonic oscillator coupled to a bosonic heat bath at zero temperature, to a squeezed state? Which ones are required for a Schrödinger cat state?

6. Write down the quantum Langevin equation for a harmonic oscillator, that is coupled to a heat bath. What is the two-time correlation function $\left\langle \hat{F}_{\text{in}}(t)\hat{F}_{\text{in}}^{\dagger}(t') \right\rangle$ of the Langevin force operator? Explain why the position power spectral density of the oscillator's position $S_{xx}(\omega)$ is asymmetric in Fourier frequency domain.

7. Explain briefly the two ways in which one can derive from a stochastic Langevin equation the equivalent Fokker Planck equation.

8. A harmonic oscillator is coupled to a bath with thermal quanta n_{th}. Write down the birth-death rates for the energy levels $|n\rangle$. How do the transition rates scale with bath thermal quanta n_{th} for a two-level system instead.

9. Explain the difference between a finite intensity and infinite intensity process in the context of a noise spectral density?

2. Part B (8pt/problem)

Please provide detailed calculation for all questions (*Estimated time: 90 min*)

Problem 1: Feedback cooling of a mechanical oscillator

In this problem, we first review some concepts about Brownian motion of a mechanical oscillator and then see how applying a feedback force results in cooling of the Brownian motion (i.e. also called 'cold' damping) of the oscillator. Consider a mechanical oscillator with mass m, spring constant of k and damping rate of Γ (Fig. **??**). The oscillator is in equilibrium with a thermal bath at temperature T. Our treatment here is completely classical.

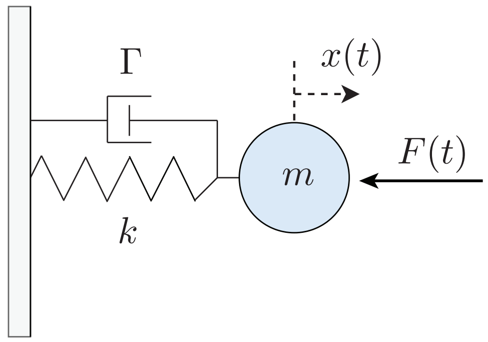

Figure 2: Mechanical oscillator

1. Suppose that the oscillator is subjected to a generic force of $F(t)$ (Fig. **??**). Write down the equation of motion for the displacement of the oscillator from its equilibrium, $x(t)$, both in time and frequency domains. Define the mechanical susceptibility as $\chi(\omega) \equiv F(\omega)/x(\omega)$ and write it down separately. (Use the mechanical frequency $\Omega = \sqrt{k/m}$ to simplify your equations.)

2. Now suppose that the only force acting on the oscillator is the thermal Langevin force, $F_{\text{th}}(t)$. Use the fluctuation-dissipation theorem and find the double-sided power spectral densities (PSD) for the thermal force, $S_{FF}^{\text{th}}(\omega)$ and the displacement $S_{xx}(\omega)$.

3. For a low-loss oscillator ($\Omega \gg \Gamma$) and at a frequency range close to the resonance ($|\omega - \Omega| \gg \Gamma$), simplify $\chi(\omega)$ and $S_{xx}(\omega)$ into form of a Lorentzian. Use this approximation for the rest of the problem.

4. Calculate the total RMS displacement fluctuations, $x_{\text{rms}}^{\text{th}} = \sqrt{\langle x^2 \rangle}$. Compare your result to what you expect from the equipartition theorem.

5. What is the expression for the Bose-Einstein thermal occupation of the mechanical mode, n_m^{th}? Estimate n_m^{th} for an oscillator with frequency of 1 MHz at room temperature.

   Now suppose that we have a way of measuring the position of the oscillator. The eye in **??** represents our detector where its physical realization is irrelevant to our problem. The measured signal by the detector is given by

$$y(t) = x(t) + x_{\text{imp}}(t) \tag{1}$$

   where $x(t)$ is the oscillator's displacement and $x_{\text{imp}}(t)$ is the noise added by the detector. For simplicity we assume that the detector noise is negligible $x_{\text{imp}}(t) \approx 0$. As shown in Fig.

??, there is a feedback mechanism that translates the measured signal into a force on the oscillator. The feedback force is given by

$$F_{\text{fb}}(t) = m\Gamma_{\text{fb}}\dot{y}(t), \tag{2}$$

where Γ_{fb} is the feedback gain that we can control both its magnitude and sign.

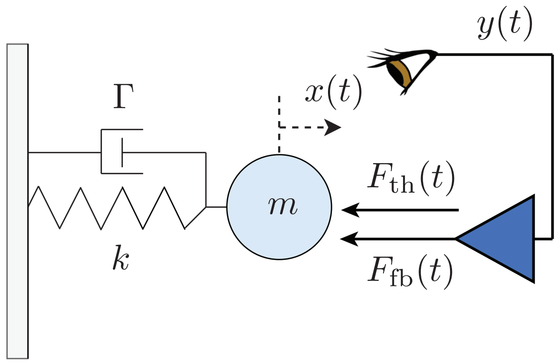

Figure 3: Mechanical oscillator subject to feedback force

6. In this scenario there are two forces acting on the oscillator: The thermal Langevin force and the feedback force. Write down the equations of motion for $x(t)$ both in time and frequency domains.

7. Now from the point of view of the Langevin force you can define a new effective susceptibility so that
$$x(\omega) = \chi_{\text{eff}}(\omega)F_{\text{th}}(\omega). \tag{3}$$
Find the expression for $\chi_{\text{eff}}(\omega)$.

8. Using Eq. **??** and your result from part 2, find the new PSD of the displacement fluctuations.

9. Calculate the new RMS displacement fluctuations, $x_{\text{rms}}^{\text{fb}} = \sqrt{\langle x^2 \rangle}$. Define an effective temperature, T_{eff}, by making an analogy with the equipartition theorem.

10. Compare T_{eff} to T for different signs of Γ_{fb}.

Problem 2: Vasicek model

In this problem, we will review a stochastic model named Vasicek model, which describes the evolution of interest rate in finance. The instantaneous interest rate r follows the stochastic differential equation:

$$dr = (\alpha - \beta \cdot r)\, dt + \sigma dW,$$

where $\beta > 0$ and dW is the Wiener increment.

1. Find the solution of the Vasicek model, i.e. obtain the expression of r explicitly in terms of time t, input noise dW and intial value r_0. (Hint: consider the function $f(t,r) = e^{\beta t} r$ and use Ito's lemma. The result can contain an integral on dW.)

2. From the solution, derive the long term mean level and long term variance of the interest rate r as $t \to \infty$.

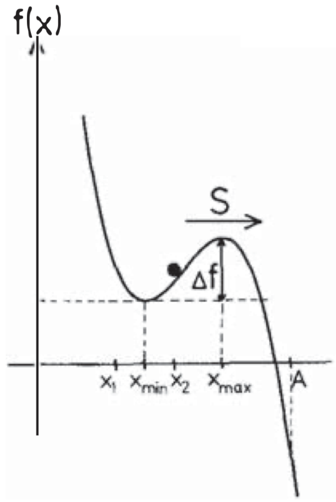

Figure 4: Potential well for calculating escape rate.

Problem 3: Proof of Kramers escape rate:

Kramers famous equation relates the escape rate ($2\pi r = \frac{1}{t_{\text{escape}}}$) of a particle to the curvature of the potential at the bottom and the top (see Fig. **??**). Assume that the potential $V(x)$ has a minimum at x_{min} and a maximum at $x_{\max}$ so that $0 < x_{\min} < x_{\max} < A$. The escape rate is given by (in the limit of a strongly damped particle and a high barrier compared to the thermal energy):

$$r = (2\pi)^{-1} \cdot \frac{D}{k_B T} \cdot \sqrt{V''(x_{\min})|V''(x_{\max})|} \cdot e^{-(V(x_{\max})-V(x_{\min}))/k_B T} \tag{4}$$

Derive this rate following these short questions.

1. First, assume that the particle is strongly damped and solve the simplified 1-D Fokker Planck equation
$$\partial_t P(x,t) = \frac{D}{k_B T}\partial_x\left(\frac{dV(x)}{dx}P(x,t)\right) + D\partial_x^2 P(x,t) \tag{5}$$
with a constant probability current ($J(x) \approx J = \text{const}$, obeys the continuity equation $\partial_t P(x,t) + \partial_x J(x) = 0$). Show that with $\Phi(x) = V(x)/k_B T$ the current satisfies:
$$J(x,t) = -De^{-\Phi(x)}\frac{d}{dx}[e^{+\Phi(x)}P(x,t)] \tag{6}$$

2. As outlined above we are treating the problem without reflecting or absorbing boundary conditions, but assume the current to be nonzero even on our boundaries. As such we can neglect the time dependence of $P(x,t)$ and $J(x,t)$ since the solution will become stationary in this case. Why does it become stationary? Obtain an expression for the current by integrating over the interval $x = x_{min}...A$. Find an expression for J and show that it obeys:
$$J = D \cdot e^{V(x_{\min})/k_B T} P(x_{\min}, t) / \left(\int_{x_{\min}}^{A} e^{V(x)/k_B T} dx\right) \tag{7}$$

3. Approximate the distribution $P(x_{\min}, t)$ by assuming the stationary distribution of the parabolic well only (i.e. such that particles cannot escape from near $x_{\min}$). This will be an excellent approximation as long as the barrier is high compared to $k_B T$.

4. Obtain an expression for the escape rate r, i.e. $J = pr$, which is defined as the ratio of the probability current J and the absolute probability p to find the particle inside the well.

5. Solve the integrals by considering the regions where the integrals contribute most and by extending the integration limits $(0..A)$ to $\pm\infty$. Use the local Taylor expansion in the potential and use $\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ to arrive at Kramers escape rate.

Problem 4: Two-level system as a spectrum analyser

Consider a two-level system whose free evolution is determined by the Hamiltonian

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z.$$

It is coupled to an external "field" $f(t)$ via a dipole-like interaction,

$$\hat{H}_{int} = \hbar g f(t)\hat{\sigma}_x.$$

The Pauli operators are given by

$$\hat{\sigma}_z = |1\rangle\langle 1| - |0\rangle\langle 0|, \tag{8}$$
$$\hat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|, \tag{9}$$

where $|0\rangle$ and $|1\rangle$ correspond to the ground and excited state respectively. We will assume that $g \ll \omega_0$, so that the system can be analyzed perturbatively.

1. Assume that the two-level system is prepared in its ground state $|0\rangle$ at time $t = 0$. Using the Schroedinger equation show that the amplitude for the system to be in its excited state at time t is given by,
$$\langle 1|\psi(t)\rangle = -ig\int_0^t f(t')e^{-i\omega_0 t'}\,dt' + \mathcal{O}(g^2).$$

2. Thus, show that the average $\mathbb{E}$ (over realizations of the possibly random "field" f) probability to find the system in the excited state is given by,
$$p_1(t) = \mathbb{E}\left[|\langle 1|\psi(t)\rangle|^2\right] \approx g^2\int_0^t e^{i\omega_0(t'-t'')}\mathbb{E}\left[f(t')f(t'')\right]\,dt'\,dt''.$$

3. Assume now that f is weak and stationary. Show that under this condition, The transition rate to the upper state, $\Gamma_1 = \dot{p}_1(t)$, takes the form,
$$\Gamma_1 \approx g^2 \cdot S_{ff}(\omega_0),$$
where $S_{ff}(\omega_0)$ is the double-sided spectral density of the field f.